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EXPANSION OF N-EXTENDED TWO-DIMENSIONAL SCALAR SUPERFIELD

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ԵՐԿՉԱՓ ՍԿԱԼՅԱՐ N -ԸՆԴԱՐՁԱԿՎԱԾ ԳԵՐԴԱՇՏԻ ՎԵՐԼՈՒԾՈՒՄԸ

Կառուցված է երկչափ սկալյար N -ընդարձակված գերդաշտի վերլուծումը անվերածելի ներկայացումների: Գտնված է գերպրոյեկտորների լրիվ բազմությունը և բերված են կապի հավասարումները, որոնց լուծումներն էլ հենց անվերածելի ներկայացումներ են: Դիտարկված են անվերածելի գերդաշտերի օրինակներ, և նույնացված է նրանց տեղն ընդհանուր սխեմայում:

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EXPANSION OF N-EXTENDED TWO-DIMENSIONAL SCALAR SUPERFIELD

The expansion of two-dimensional N-extended scalar superfield into irreducible representations is constructed. The full set of superprojectors is found, coupling equations separating irreducible representations are written and their solutions are found. Examples of irreducible superfields are considered and their place in the general scheme is identified.

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РАЗЛОЖЕНИЕ N - РАСШИРЕННОГО ДВУХМЕРНОГО СКАЛЯРНОГО СУПЕРПОЛЯ

Построено разложение двухмерного N -расширенного скалярного суперполя на неприводимые представления. Найден полный набор суперпроекторов, выписаны уравнения связи, выделяющие неприводимые представления, и найдены их решения. Рассмотрены примеры неприводимых суперполей и идентифицировано их место в общей схеме.

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1. The development of supersymmetry and construction of new models always progress in parallel directions: in the component language and in terms of superfields. The component language is more thrifty regarding the composition of fields, as it includes mainly the set of physical fields, but the expressions for actions, equations of motion, etc. are obtained quite cumbersome. Formulation in terms of superfields is more convenient which though using many (sometimes a great number of) auxiliary fields, combines them in one object (superfield), has an explicit supersymmetry off the mass shell, simplifies calculations on perturbation theory and, consequently, the problem of cancellation of divergences, etc. In this connection there arises the question about determination of the structure of superfields and the construction of superfields corresponding to the irreducible representations of the supersymmetry algebra. Despite the voluminous literature devoted to this problem (see, e.g., [1]), one cannot say that the problem has an explicit and final solution for all cases.

In the present work such problem is solved for a supersymmetry in two-dimensional space-time. Namely, the expansion of superfields into irreducible representations of the supersymmetry algebra is constructed for any N in the sense that the full set of projectors on the irreducible representations is constructed, the corresponding coupling equations separating out irreducible representations are found, and the solution of these equations is explicitly found. The consideration is limitted by the case of scalar superfields. The superfields with indices (Lorentz and the supersymmetry algebra automorphism groups indices) will be considered separately. The standard method of induced representations (see [2]) is used in the solution.

The paper is arranged as follows. In section 2 indications are introduced, the supersymmetry algebras are written and the structure of irreducible representations is described. In section 3 , based on the results from the section 2, there are constructed projectors, coupling equations are written and the solution of these equations is shown for irreducible representations contained in the scalar superfield. It turns out that expansion of the superfield into irreducible superfields can be done not in a single way (in a sense all the irreducible superfield representations appear to be equivalent). In section 4 two concrete realizations of the general method of the superfield expansion into irreducible components are presented. In section 5 there are considered examples of superfields obtained in different ways: superfield obtained as a result of reduction of four-dimensional N=1 and N=2 superfields into two-dimensional space-time, and also the so--called twisted superfields considered in ref.[3]; their place in the general scheme is identified.

2. Let us consider the N-extended S_N algebra of the supersymmetry which is realized in the space of scalar superfields $\Phi(x,\theta)$; x^{μ} ($\mu = 0, 1$) are the coordinates of the

two-dimensional space-time, θ_α^a ($\alpha = 1,2$; $a = 1, \dots, N$) is the set of N Majorana spinors. We shall use the following choice of metrics and γ-matrices:

$$g_{\mu\nu} = (+,-), \quad \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad C_{\alpha\beta} C^{\beta\gamma} = -\delta_\alpha^\gamma ,$$

The derivatives with respect to Grassmanian variables are the left ones.

At such choice the Majorana condition is the reality θ_α^a: $(\theta_\alpha^a)^+ = \theta_\alpha^a$. The S_N algebra is generated by the generators of the Poincare group

$$P_\mu = i\partial_\mu ; \qquad L_{\mu\nu} = -\frac{1}{2}\left(\bar{\theta}^a \sigma_{\mu\nu} \frac{\partial}{\partial \bar{\theta}^a}\right) + i\,(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

and by the generators of the supersymmetry $S^{\alpha,a} = \partial/\partial\theta_\alpha^a + (i/2)(C\gamma^\mu)^{\alpha\beta}\theta_\beta^a \partial_\mu$ which satisfy the anticommutation relations

$$\{ S^{\alpha,a}, S^{\beta,b} \} = i\delta^{ab} (C\gamma^\mu)^{\alpha\beta} \partial_\mu$$

At Lorentz transformations θ_α^a is transformed as follows:

$$\delta\theta^a = [i\omega^{\mu\nu} L_{\mu\nu}, \theta^a] = i\omega(\gamma^5 \theta^a),$$

where $\omega^{\mu\nu} = \varepsilon^{\mu\nu}\omega$, $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, $\varepsilon^{01} = 1$. It is seen that θ_1^a and θ_2^a are transformed independently during Lorentz transformations, which is due to the Abelian character of the Lorentz group in two-dimensional space-time.

Let us also introduce the extended $\not{S}_N$ supersymmetry algebra which together with the generators of the S_N algebra also contains the generators $D^{\alpha,a}$ which have anticommutation relations:

$$\{D^{\alpha,a}, D^{\beta,b}\} = -i\delta^{ab}(C\gamma^{\mu})^{\alpha\beta}\partial_{\mu}, \quad \{D^{\alpha,a}, S^{\beta,b}\} = 0,$$

$$D^{\alpha,a} = \frac{\partial}{\partial\theta^{a}_{\alpha}} - \frac{i}{2}(C\gamma^{\mu})^{\alpha\beta}\theta^{a}_{\beta}\partial_{\mu};$$

The irreducible representations will be constructed by the method of induced ones [2]. Here as an irreducible representation is ment to be that in which the eigenvalues of all Casimir operators, except for P^{i}, are defined.

The generators $S^{\alpha,a}$ form a Clifford algebra with an even number, 2N, of generatrices. As all the irreducible representations of the Lorentz group are one-dimensional in a two-dimensional space-time, and the irreducible representation of the Clifford algebra with 2N generatrices has dimension 2^N, then it is clear that any irreducible representation of the S_N algebra contains 2^N component fields: 2^{N-1} Bose and 2^{N-1} Fermi ones.

Note, that the general scalar superfield forms the irreducible representation of the $\mathcal{S}_N$ algebra [1]. This follows from the fact that the generators $S^{\alpha,a}$ and $D^{\alpha,a}$ together form the Clifford algebra with 4N generatrices and hence, the irreducible representation of the $\mathcal{S}_N$ algebra contains $2^{2N-1} + 2^{2N-1} = 2^{2N}$ component fields; in expansion of the general scalar superfield $\phi(x,\theta)$ in θ^{a}_{α} just 2^{2N-1} Bose and 2^{2N-1} Fermi fields are present. Thus, our task is to expand the 2^{2N}-dimensional irreducible representation of the $\mathcal{S}_N$ algebra into 2^N-dimensional ones of the S_N algebra. The number of the latters will apparently be 2^N.

We'll act the following way to solve this problem. Instead of operators $\mathcal{S}^{\alpha,a}$ introduce new ones

$$s^{\alpha,a} = \mathcal{A}^{\alpha a/\beta b}\mathcal{S}^{\beta,b}, \qquad (1)$$

where $\mathcal{A}^{\alpha a/\beta b}$ is a nonsingular numerical matrix, and the operators $s^{\alpha,a}$ satisfy the anticommutation relations

$$\{s^{\alpha,a}, s^{\beta,b}\} = \delta^{ab}(\sigma_1)^{\alpha\beta}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (2)$$

Analogously, introduce new operators $d^{\alpha,a}$:

$$d^{\alpha,a} = \mathcal{A}^{\alpha a/\beta b}\mathcal{D}^{\beta,b},$$

for which the anticommutation relations have the following form:

$$\{d^{\alpha,a}, d^{\beta,b}\} = -\delta^{ab}(\sigma_1)^{\alpha\beta}, \quad \{d^{\alpha,a}, s^{\beta,b}\} = 0. \qquad (3)$$

It is clear that there is an infinitely great number of such matrices $\mathcal{A}^{\alpha a/\beta b}$. Also assume, that the matrix $\mathcal{A}^{\alpha a/\beta b}$ is such, that the operators $s^{\alpha,a}$ and $d^{\alpha,a}$ form the Lorentz group representation at any fixed values of α and a. We shall consider the operators $s^{1,a}$ and $d^{1,a}$ as ones of annihilation, and the operators $s^{2,a}$ and $d^{2,a}$ - as ones of production. As the superfield $\Phi(x,\theta)$ forms the irreducible representation of the $\mathcal{S}_N$ algebra, there exists the only vector $|0\rangle$ (Clifford vacuum) satisfying the condition

$$s^{1,a}|0\rangle = d^{1,a}|0\rangle = 0$$

for all values of a. The basis states of the representation are obtained by acting of all possible products of the operators $s^{2,a}$ and $d^{2,a}$ on $|0>$. There are exactly 2^{2N} such vectors. To expand this representation of the $\mathcal{S}_N$ algebra in those of the S_N algebra, note, that the vector

$$|m_1, \dots, m_K> = d^{2,m_1} d^{2,m_2} \dots d^{2,m_K} |0>, \quad m_1, m_2, \dots = 1, \dots, N$$

in virtue of anticommutation operators $s^{\alpha,a}$ and $d^{\alpha,a}$ is annihilated by the operators $s^{1,a}$:

$$s^{1,a} | m_1, \dots, m_K > = 0 .$$

So, it can be considered as a vacuum state of the $s^{\alpha,a}$ operators' algebra representation. The basis of the irreducible representation of the $s^{\alpha,a}$ operators' algebra (hence, that of the $\mathcal{S}^{\alpha,a}$ operators algebra too) is obtained to be the action on the vector $|m_1, \dots, m_K>$ of all possible products of the operators $s^{2,a}$. The dimension of this representation is 2^N. With respect to the assumed Lorentz properties of operators $s^{\alpha,a}$ and $d^{\alpha,a}$ this representation is also an irreducible representation of the S_N algebra. Thus, each vector $|m_1, \dots, m_K>$ generates an irreducible representation of the S_N algebra (of dimension 2^N). So far as there are 2^N such vectors, we obtain 2^N irreducible representations of the S_N algebra each one of which has 2^N dimensions. Dimension of the space formed by the direct sum of spaces of such representations is 2^{2N} and hence, it coincides with the space of the component fields which is described by the scalar superfield $\Phi(x, \theta)$. So, construction of the S_N algebra irreducible representations contained in the superfield $\Phi(x, \theta)$

is realized as follows: the Clifford vacuum $|0\rangle$ is found, 2^N vacuum vectors $|m_1,\ldots,m_K\rangle$ are constructed by it, and then through each of such vectors an irreducible representation of the S_N algebra is constructed.

3. Now we shall describe irreducible superfields which comply with the irreducible representations of the S_N algebra obtained in the previous section. The superfield complying with the vacuum vector $|m_1,\ldots,m_K\rangle$ will be denoted by $\phi_{m_1,\ldots,m_K}$. It follows from (2) and (3) properties of the operators S and d that the superfield $\phi_{m_1,\ldots,m_K}$ satisfies the conditions (they are sometimes called coupling equations)

$$
\begin{aligned}
&d^{2,m}\phi_{m_1,\ldots,m_K}=0, \qquad m\in\{m_1,\ldots,m_K\} \\
&d^{1,m}\phi_{m_1,\ldots,m_K}=0, \qquad m\notin\{m_1,\ldots,m_K\}
\end{aligned}
\tag{4}
$$

Inversely, the set of conditions (4) defines the irreducible representation. Really, since the operator d anticommutates with the S operators, the solution of (4) forms the S_N algebra representation. This is not a trivial representation, as it contains at least one constructed by the vacuum vector $|m_1,\ldots,m_K\rangle$. That is why its dimensions $d_{m_1,\ldots,m_K}$ must be no less than 2^N: $d_{m_1,\ldots,m_K}\geqslant 2^N$. There are 2^N different sets of equations of (4) type (in accordance with the total number of different vacuum vectors) and, consequently, 2^N different superfields $\phi_{m_1,\ldots,m_K}$. Since the sum of dimensions of all the superfields (by the superfield dimensions we mean the number of independent field components equal to the dimensions of the corresponding representation) can not exceed 2^{2N}, we conclude that

$$d_{m_1,\dots,m_K} = 2^N .$$

Thus, each one of 2^N superfields $\Phi_{m_1,\dots,m_K}$ complies with the nontrivial irreducible representation of the S_N algebra.

It is also easy to construct the full set of superprojectors on the superfields $\Phi_{m_1,\dots,m_K}$. Formally, to construct them, we'll proceed as follows. Consider a system of N commutation relations (see (3))

$$-\left(d^{1,m_1} d^{2,m_1} + d^{2,m_1} d^{1,m_1}\right) = 1$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$-\left(d^{1,m_N} d^{2,m_N} + d^{2,m_N} d^{1,m_N}\right) = 1$$

(among the values of $m_1,\dots,m_N$ there are no repeating ones) and multiply their right hand sides and left hand sides, correspondingly. We obtain

$$1 = (-1)^N \prod_{i=1}^{N} \left(d^{1,m_i} d^{2,m_i} + d^{2,m_i} d^{1,m_i}\right) =$$

$$= P + \sum_{m_1} P^{m_1} + \sum_{m_1<m_2} P^{m_1 m_2} + \dots + \sum_{m_1<m_2<\dots<m_K} P^{m_1\cdots m_K} + \dots + P^{m_1\cdots m_N}, \qquad (5)$$

where the following denotations are introduced:

$$P = (-1)^N \left(d^{1,m_1} d^{2,m_1}\right) \dots \left(d^{1,m_N} d^{2,m_N}\right),$$

$$P^{m_1\cdots m_K} = (-1)^N \left(d^{2,m_1} d^{1,m_1}\right) \cdots \left(d^{2,m_K} d^{1,m_K}\right)\left(d^{1,m_{K+1}} d^{2,m_{K+1}}\right) \dots \left(d^{1,m_N} d^{2,m_N}\right)$$

$$P^{m_1\cdots m_N} = (-1)^N \left(d^{2,m_1} d^{1,m_1}\right) \cdots \left(d^{2,m_N} d^{1,m_N}\right) \qquad (6)$$

Note that in the right hand side of (5) there are 2^N summands. The common term in this sum (see (6)) has indices which numerate the cofactors in which the production operator d^2 stands first (by cofactor we mean a pair of operators in the parentheses in (6)). It is obvious that all the cofactors commutate with each other.

It is easy to ensure that the operators $P^{m_1 m_2 \cdots}$, apart from the condition (5) - the equality of the sum of all the operators to unity - also satisfy the conditions

$$P^{m_1\cdots m_K} P^{m'_1\cdots m'_{K'}} = \begin{cases} P^{m_1\cdots m_K} & , \{m_1,\cdots,m_K\} = \{m'_1,\dots,m'_{K'}\} \\ 0 & , \{m_1,\dots,m_K\} \neq \{m'_1,\dots,m'_{K'}\} \end{cases}$$

$$d^{2,m} P^{m_1\cdots m_K} = 0 , \qquad m \in \{m_1,\dots,m_K\}$$

$$d^{1,m} P^{m_1\cdots m_K} = 0 , \qquad m \notin \{m_1,\dots,m_K\}$$

Hence, the operators $P^{m_1\cdots m_K}$ represent themselves the set of 2^N superprojectors sought for; the result of their action on the general superfield is expressed by

$$P^{m_1\cdots m_K} \Phi = \Phi_{m_1,\cdots,m_K}$$

In the end of this section we shall show the possibility for an explicit solution of the coupling equations (4). Any of the coupling equations has the form

$$d\Phi = 0, \tag{7}$$

where d is an operator having the form

$$d = \tilde{\theta} + \partial,$$

$\tilde{\theta} = \sum p^{\alpha,a} \theta^a_\alpha$, $\partial = \sum q^{\alpha,a} (\partial / \partial \theta^a_\alpha)$; the factors $p^{\alpha,a}$ and $q^{\alpha,a}$ may depend on the derivatives of spatial coordinates. It follows from the condition $(d)^2 = 0$ that $\{\tilde{\theta}, \partial\} = 0$. It means that one can linearly transform the coordinates $\theta^a_\alpha \equiv \theta_\beta \to \theta'_\beta$ so, that $\tilde{\theta} = \theta'_1$, $\partial = \partial / \partial \theta'_2$ are satisfied. Then the solution of the equation (7) can be represented as

$$\Phi = \exp\{\theta'_1 \theta'_2\} \Phi_1 , \qquad \frac{\partial}{\partial \theta'_2} \Phi_1 = 0 . \tag{8}$$

Solving thus the first one of the coupling equations (4) and substituting the solution of type (8) into the second coupling equation, we again obtain an equation of type (7) for Φ_1 . The solution for Φ_1 , in its turn, is presented in the form of (8) and is substituted into the third coupling equation and so on. Finally, the solution of the coupling equations (4) looks like

$$\Phi_{m_1,\dots,m_K} = \exp\left\{ \frac{1}{2} \theta^a_\alpha B^{\alpha a/\beta b} \theta^b_\beta \right\} \bar{\Phi}_{m_1,\dots,m_K}$$

where $\bar{\Phi}_{m_1,\dots,m_K}$ is a superfield of general form, but depending only on some N independent linear combinations of $2N$ variables θ^a_α ; $B^{\alpha a/\beta b}$ is some matrix.

Note, that it follows from (8) that each one of the conditions (4) twice reduces the number of independent field components so, that the number of independent components of the solution of the set of N equations (4) (equal to the number of independent components of the superfield $\bar{\Phi}_{m_1,\dots,m_K}$) is equal to $2^{2N}: 2^N = 2^N$, i.e. $\Phi_{m_1,\dots,m_K}$ is a nontrivial irreducible representation of the S_N algebra.

In section 2 it was noted that there is some freedom in the choice of the matrix $A^{\alpha a/\beta b}$ by means of which the operators $S^{\alpha,a}$ and $d^{\alpha,a}$ were introduced. The other matrix $A'^{\alpha a/\beta b}$ proving the satisfaction of the relations (2) and (3), apparently, generates another expansion of the superfield Φ into irreducible representations. The S_N algebra irreducible representations generated by the matrix $A'^{\alpha a/\beta b}$ are the linear superposition of the irreducible representations generated by the matrix $A^{\alpha a/\beta b}$. It means that among the irreducible representations generated by the matrix $A^{\alpha a/\beta b}$ must be equivalent ones.

We'll now show, that it is really so. Consider the sets $\{m_1,\dots,m_K\}$ and $\{n_1,\dots,n_\ell\}$ and the superfield $\Phi_{m_1,\dots,m_K}$. Construct the superfield

$$\Phi' = \prod_{m_i\notin\{n_1,\dots,n_\ell\}} d^{1,m_i} \prod_{n_j\notin\{m_1,\dots,m_K\}} d^{2,n_j} \quad \Phi_{m_1,\dots,m_K}.$$

It satisfies the conditions

$$\begin{aligned} d^{2,m}\Phi' &= 0\,, \qquad m\in\{n_1,\dots,n_\ell\} \\ d^{1,m}\Phi' &= 0\,, \qquad m\notin\{n_1,\dots,n_\ell\}\,, \end{aligned} \tag{9}$$

coinciding with those for the superfield $\Phi_{n_1,\dots,n_\ell}$. It means that the superfields $\Phi_{m_1,\dots,m_K}$ and $\Phi_{n_1,\dots,n_\ell}$ are equivalent irreducible representations of the $P_\mu, S^{\alpha,a}$ generators' algebra. The superfields Φ' and $\Phi_{n_1,\dots,n_e}$ may be different representations of the Lorentz group. If the weight factor M is possible to choose so that the superfields $\Phi'' = M\Phi'$ and $\Phi_{n_1,\dots,n_e}$ will relate to the same representation of the Lorentz group and the field Φ'' will still satisfy the conditions (9), then the superfields $\Phi_{m_1,\dots,m_k}$ and $\Phi_{n_1,\dots,n_e}$ turn out to be equivalent irreducible representations of the S_N algebra (the weight factor M, for instance, can be constructed from the product of different powers of operators $(\partial_0 - \partial_1)$ and $(\partial_0 + \partial_1)$.

4. Here will be presented two examples of the choice of the matrix $A^{\alpha a/\beta b}$ and construction of the corresponding expansions of the superfields. As the first example consider the matrix

$$A^{\alpha a/\beta b} = \delta^{ab} \begin{pmatrix} \frac{1}{2\sqrt{i\partial v}} & \frac{i}{2\sqrt{i\partial u}} \\ \frac{1}{2\sqrt{i\partial v}} & -\frac{i}{2\sqrt{i\partial u}} \end{pmatrix},$$

$$\partial_u = \frac{1}{2}(\partial_0 - \partial_1) \quad , \qquad \partial_v = \frac{1}{2}(\partial_0 + \partial_1) \, ,$$

which realizes the rotation in the space of the "usual" spin, not affecting the "isotopic" index. Its explicit form directly follows from the expressions for $S^{\alpha,a}$ and $D^{\alpha,a}$ and from the condition that $s^{\alpha,a}$ and $d^{\alpha,a}$ satisfy the anticommutation relations

$$\{ s^{1,a}, s^{2,b} \} = \delta^{ab} \quad , \qquad \{ d^{1,a}, d^{2,b} \} = -\delta^{ab}$$

(all the other anticommutators are equal to zero). Let us restrict ourselves to the case with N=1; the generalization in case of N > 1 is obvious.

The operators S^{α} and d^{α} are equal to

$$S^1 = \frac{1}{2\sqrt{i\partial v}}\left(\frac{\partial}{\partial\theta_1} + i\theta_1\partial v\right) + \frac{i}{2\sqrt{i\partial u}}\left(\frac{\partial}{\partial\theta_2} + i\theta_2\partial u\right),$$

$$S^2 = \frac{1}{2\sqrt{i\partial v}}\left(\frac{\partial}{\partial\theta_1} + i\theta_1\partial v\right) - \frac{i}{2\sqrt{i\partial u}}\left(\frac{\partial}{\partial\theta_2} + i\theta_2\partial u\right)$$

$$d^1 = \frac{1}{2\sqrt{i\partial v}}\left(\frac{\partial}{\partial\theta_1} - i\theta_1\partial v\right) + \frac{i}{2\sqrt{i\partial u}}\left(\frac{\partial}{\partial\theta_2} - i\theta_2\partial u\right)$$

$$d^2 = \frac{1}{2\sqrt{i\partial v}}\left(\frac{\partial}{\partial\theta_1} - i\theta_1\partial v\right) - \frac{i}{2\sqrt{i\partial u}}\left(\frac{\partial}{\partial\theta_2} - i\theta_2\partial u\right).$$

According to the general theory the superfield $\Phi(x,\theta_1,\theta_2)$ expands into a sum of two irreducible representations of the S_1 algebra:

$$\Phi = \Phi_0 + \Phi_1 \;; \qquad d^1\Phi_0 = 0, \qquad d^2\Phi_1 = 0\,.$$

The solution of these equations has the form

$$\Phi_0 = e^{-\frac{1}{2}\theta_1\theta_2\sqrt{\Box}}\varphi_0(x,\theta_1\sqrt{i\partial v} + i\theta_2\sqrt{i\partial u}), \quad \Phi_1 = e^{\frac{1}{2}\theta_1\theta_2\sqrt{\Box}}\varphi_1(x,\theta_1\sqrt{i\partial v} - i\theta_2\sqrt{i\partial u})$$

The corresponding superprojectors are

$$P_0 = -d^1d^2 = \frac{1}{2} + \frac{1}{\sqrt{\Box}}\left(\frac{\partial}{\partial\theta_1} - i\theta_1\partial v\right)\left(\frac{\partial}{\partial\theta_2} - i\theta_2\partial u\right)$$

$$P_1 = -d^2d^1 = \frac{1}{2} - \frac{1}{\sqrt{\Box}}\left(\frac{\partial}{\partial\theta_1} - i\theta_1\partial v\right)\left(\frac{\partial}{\partial\theta_2} - i\theta_2\partial u\right)$$

Then, representing the expansion of Φ , Φ_0 and Φ_1 superfields as

$$\Phi = A + \theta_1\Psi_1 + \theta_2\Psi_2 + \theta_1\theta_2 F$$

$$\Phi_0 = A_0 + (\theta_1\sqrt{i\partial v} + i\theta_2\sqrt{i\partial u})\lambda_0 - \frac{1}{2}\theta_1\theta_2\sqrt{\Box}A_0$$

$$\Phi_1 = A_1 + (\theta_1\sqrt{i\partial v} - i\theta_2\sqrt{i\partial u})\lambda_1 + \frac{1}{2}\theta_1\theta_2\sqrt{\Box}A_1$$

we find the connection between their components:

$$A_0 = \frac{1}{2}A - \frac{1}{\sqrt{\Box}}F\,, \qquad A_1 = \frac{1}{2}A + \frac{1}{\sqrt{\Box}}F\,,$$

$$\lambda_0 = \frac{1}{2\sqrt{i\partial v}}\Psi_1 - \frac{i}{2\sqrt{i\partial u}}\Psi_2\,, \qquad \lambda_1 = \frac{1}{2\sqrt{i\partial v}}\Psi_1 + \frac{i}{2\sqrt{i\partial u}}\Psi_2$$

The drawback of such an expansion of N=1 scalar superfield into irreducible ones is in its nonlocal character which shows itself in the presence of $\sqrt{\Box}$, $\sqrt{i\partial_u}$, $\sqrt{i\partial_v}$ operators.

As the second example consider the case when the matrix $A^{\alpha a/\beta \ell}$ realizes rotation in the space of the isotopic spin. Let N be an even number: N=2l. Let us group the multitude of 2l Majorana spinors θ into two ones: $\theta_\alpha^1, \dots, \theta_\alpha^{2\ell} = \{\theta_\alpha^a, \theta_\alpha^{a+\ell}\}$, $a = 1, \dots, \ell$

Introduce new coordinates

$$\eta_A = \frac{1}{\sqrt{2}}(\theta_\alpha^a + i\theta_\alpha^{a+\ell}), \quad \overset{+}{\eta}_A = \frac{1}{\sqrt{2}}(\theta_\alpha - i\theta_\alpha^{a+\ell}), \qquad A = (a, \alpha)$$

and operators

$$s_1^A = \frac{1}{\sqrt{2}}(S^{\alpha,a} - iS^{\alpha,a+\ell}), \quad s_2^A \equiv \overset{+}{s}{}_1^A = \frac{1}{\sqrt{2}}(S^{\alpha,a} + iS^{\alpha,a+\ell})$$

$$d_1^A = \frac{1}{\sqrt{2}}(D^{\alpha,a} - iD^{\alpha,a+\ell}), \quad d_2^A \equiv \overset{+}{d}{}_1^A = \frac{1}{\sqrt{2}}(D^{\alpha,a} + iD^{\alpha,a+\ell})$$

In terms of η and $\overset{+}{\eta}$ these operators have the form

$$
\begin{aligned}
S_1^A &= \frac{\partial}{\partial \eta_A} + \frac{i}{2}(C\gamma^\mu)^{AA'} \overset{+}{\eta}_{A'} \partial_\mu , \quad S_2^A = \frac{\partial}{\partial \overset{+}{\eta}_A} + \frac{i}{2}(C\gamma^\mu)^{AA'} \eta_{A'} \partial_\mu \\
d_1^A &= \frac{\partial}{\partial \eta_A} - \frac{i}{2}(C\gamma^\mu)^{AA'} \overset{+}{\eta}_{A'} \partial_\mu , \quad d_2^A = \frac{\partial}{\partial \overset{+}{\eta}_A} - \frac{i}{2}(C\gamma^\mu)^{AA'} \eta_{A'} \partial_\mu
\end{aligned} \tag{IO}
$$

$$
(C\gamma^\mu)^{AA'} \equiv \delta^{aa'} (C\gamma^\mu)^{\alpha\alpha'}
$$

The operators S and d have the following anticommutation relations

$$
\{S_1^A, S_1^{A'}\} = \{S_2^A, S_2^{A'}\} = 0 ; \quad \{S_1^A, S_2^{A'}\} = i\delta^{aa'} (C\gamma^\mu)^{\alpha\alpha'} \partial_\mu
$$

$$
\{d_1^A, d_1^{A'}\} = \{d_2^A, d_2^{A'}\} = 0 ; \quad \{d_1^A, d_2^{A'}\} = -i\delta^{aa'} (C\gamma^\mu)^{\alpha\alpha'} \partial_\mu
$$

$$
\{S_i^A, d_j^{A'}\} = 0 , \quad i, j = 1, 2.
$$

It is seen from these relations that the operators S_1^A and S_2^A (d_1^A and d_2^A) in virtue of diagonality of the matrix $(C\gamma^\mu)\partial_\mu$, can be identified with the annihilation and production operators, respectively.

Let us in this example too restrict ourselves to the simplest case of N=2 (l=1). Only note, that in the general case of arbitrary l the expressions for the superprojectors on the irreducible superfields coincide with those constructed by the formulae from ref.[4] .

The index A has only two values at l=1: $A = \alpha = 1,2$. According to the general theory, the scalar superfield Φ expands into four irreducible superfields:

$$
\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_{12} ,
$$

satisfying the coupling equations

$$
\begin{aligned}
&d_1^1 \Phi_0 = d_1^2 \Phi_0 = 0 \,, \qquad d_2^2 \Phi_2 = d_1^1 \Phi_2 = 0, \\
&d_2^1 \Phi_1 = d_1^2 \Phi_1 = 0 \,, \qquad d_2^1 \Phi_{12} = d_2^2 \Phi_{12} = 0.
\end{aligned} \tag{11}
$$

The solutions of these equations are

$$
\Phi_0(x,\eta,\overset{+}{\eta}) = e^{-\frac{i}{2}\bar{\eta}\gamma^\mu\eta\partial_\mu}\varphi_0(x,\overset{+}{\eta}), \quad \Phi_1(x,\eta,\overset{+}{\eta}) = e^{\frac{i}{2}\bar{\eta}\gamma^\mu\gamma^5\eta\partial_\mu}\varphi_1(x,\eta_1,\overset{+}{\eta}_2) \tag{12}
$$

$$
\Phi_2(x,\eta,\overset{+}{\eta}) = e^{-\frac{i}{2}\bar{\eta}\gamma^\mu\gamma^5\eta\partial_\mu}\varphi_2(x,\eta_2,\overset{+}{\eta}_1), \; \Phi_{12}(x,\eta,\overset{+}{\eta}) = e^{\frac{i}{2}\bar{\eta}\gamma^\mu\eta\partial_\mu}\varphi_{12}(x,\eta)
$$

Each of these four irreducible superfields has $2+2$ independent component fields.

5. In this section we shall consider some of two-dimensional superfields met in the literature and determine their place in the general scheme. As the first example consider the reduction of $N=1$, $d=4$ scalar superfield $\Phi^{(4)}(x,\theta)$ to $N=2$, $d=2$ scalar superfield $\Phi^{(2)}(x,\theta)$ (the indices (2) and (4) designate dimensions of spaces where the superfields are determined). In a four-dimensional space $\Phi^{(4)}$ expands into four irreducible representations $\Phi_0^{(4)}$, $\Phi_1^{(4)}$, $\Phi_2^{(4)}$, $\Phi_{12}^{(4)}$ of the algebra of P_μ, D_α, $\bar{D}_{\dot\alpha}$ operators determined by conditions

$$
D_\alpha \Phi_0^{(4)} = 0 \,, \qquad D_1 \Phi_2^{(4)} = \bar{D}_{\dot 2} \Phi_2^{(4)} = 0 \,,
$$

$$
D_2 \Phi_1^{(4)} = \bar{D}_{\dot 1} \Phi_1^{(4)} = 0 \,, \qquad \bar{D}_{\dot\alpha} \Phi_{12}^{(4)} = 0
$$

$$
D_\alpha = -\frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\sigma^\mu)_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_\mu \,, \quad \bar{D}_{\dot\alpha} = \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} - \frac{i}{2}\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu
$$

The fields $\Phi_0^{(4)}$ and $\Phi_{12}^{(4)}$ are ordinary chiral and antichiral superfields. The superfields $\Phi_1^{(4)}$ and $\Phi_2^{(4)}$ do not singly form Lorentz algebra representations, but only the superfield $\Phi_1^{(4)} + \Phi_2^{(4)}$ is the irreducible representation of the supersymmetry algebra known as vector superfield. Now let us reduce to a two-dimensional space. We shall assume that the fields $\Phi^{(4)}$ are independent on X^1, X^2; identify (X^0, X^3) of the four-dimensional space-time with (X^0, X^1) of the two-dimensional one. The transition from θ_α and $\bar{\theta}_{\dot{\alpha}}$ spinors to η_α and $\overset{+}{\eta}_\alpha$ ones in the two-dimensional space realize by

$$\theta_\alpha = (\sigma_2 \eta)_\alpha = \eta^\alpha, \qquad \bar{\theta}_{\dot{\alpha}} = -(\sigma_2 \overset{+}{\eta})_\alpha = \overset{+}{\eta}{}^\alpha, \qquad \bar{\theta}^{\dot{\alpha}} = \overset{+}{\eta}_\alpha$$

Here we obtain correspondences $D_\alpha \to -d_1^\alpha$, $\bar{D}_{\dot{\alpha}} \to d_2^\alpha$ (d_1^α and d_2^α are given by (10) for $A = \alpha = 1,2$) which, in their turn, bring to the following correspondences for the fields: $\Phi_0^{(4)} \to \Phi_0^{(2)}$, $\Phi_1^{(4)} \to \Phi_1^{(2)}$, $\Phi_2^{(4)} \to \Phi_2^{(2)}$, $\Phi_{12}^{(4)} \to \Phi_{12}^{(2)}$, where $\Phi^{(2)}$ exactly are solutions for eqs.(11). Note, that the transverse vector field $A_\mu^{(4)} (\partial^\mu A_\mu^{(4)} = 0)$ in the four-dimensional space-time has natural reduction determined from the correspondence $\Phi_1^{(4)} + \Phi_2^{(4)} \to \Phi_1^{(2)} + \Phi_2^{(2)}$. Namely, $A_\mu^{(2)} = (A_0^{(4)}, A_3^{(4)})$; $\partial^\mu A_\mu^{(2)} = 0$; here $A_\mu^{(2)} = \varepsilon_{\mu\nu} \partial^\nu (\varphi_1 - \varphi_2)$, where φ_1, φ_2 are the first components of the superfields $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$, respectively.

As the second example consider the so-called twisted chiral N=2, d=2 superfield χ, considered in ref.[3]. It is described by (in denotations of ref.[3])

$$D_+ \chi = \bar{D}_- \chi = 0 .$$

When θ_α , D_α , $\bar{D}_\alpha$ from ref.[3] be correspondingly identified with η_α , d_1^α , d_2^α , it is seen that $\chi = \Phi_2$ (see (11), (12)).

As a final example consider N=4, d=2 twisted multiplet mentioned in this work. It is characterized by (we directly write in our denotations):

$$d_2^{(a,\alpha)}\Phi = 0, \qquad d_1^{(a,1)}\chi = d_2^{(a,2)}\chi = 0,$$

$$(d_1^{(2,1)} + i d_2^{(1,1)})(\bar{\Phi} + \bar{\chi}) = 0, \quad (d_1^{(2,2)} - i d_2^{(1,2)})(\Phi + \bar{\chi}) = 0,$$

where we identified $D_{a\alpha}$, $\bar{D}^a_\alpha$ from ref.[3] with $d_1^{(a,\alpha)}$ $d_2^{(a,\alpha)}$, respectively.

With account of

$$\{d_1^{(a,1)}, d_2^{(b,1)}\} = \delta^{ab} i(\partial_0 + \partial_1), \quad \{d_1^{(a,2)}, d_2^{(b,2)}\} = \delta^{ab} i(\partial_0 - \partial_1)$$

from the last two equations we obtain

$$\bar{\chi} = -\frac{1}{\partial_0 + \partial_1} d_2^{(2,1)} d_2^{(1,1)} \bar{\Phi}, \qquad \Phi = \frac{1}{\partial_0 - \partial_1} d_2^{(2,2)} d_2^{(1,2)} \bar{\chi},$$

It follows, that χ is a subordinate superfield expressed in terms of Φ , and that Φ satisfies the condition of being "Hermitian":

$$\Phi = \frac{1}{\Box}(d_2^{(2,2)} d_2^{(2,1)})(d_2^{(1,2)} d_2^{(1,1)})\bar{\Phi} \equiv -\frac{1}{4\Box}(d_2)^4 \bar{\Phi} \tag{13}$$

Note, that such condition of being Hermitian is possible, since in accordance with section 2 the superfield $(d_2)^4 \bar{\Phi}$ forms the same representation of the algebra of operators P_μ , S_α

as Φ does, and the operator $(d_2)^4/\Box$ is a Lorentz-invariant one. Finally, N=4, d=2 twisted chiral superfield is the "double-chiral" superfield $\Phi_{(1,1)(1,2)(2,1)(2,2)}$ which satisfies the condition of reality (13). In such a form this supermultiplet was introduced and used for construction of the finite σ-model in ref.[5], where it was obtained as a result of reduction of the double-chiral N=2, d=4 superfield satisfying the condition of reality.

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РАЗЛОЖЕНИЕ N - РАСШИРЕННОГО ДВУХМЕРНОГО СКАЛЯРНОГО СУПЕРПОЛЯ

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