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BRST QUANTIZATION OF HAMILTONIAN SYSTEMS WITH SECOND-CLASS CONSTRAINTS

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BRST QUANTIZATION OF HAMILTONIAN SYSTEMS WITH SECOND-CLASS CONSTRAINTS

In the present work a method is proposed, which reduces the Hamiltonian dynamical systems with irreducible second-class constraints to the ones with first-class constraints.

Yerevan Physics Institute
Yerevan 1988

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БРСТ КВАНТОВАНИЕ ГАМИЛЬТОНОВЫХ СИСТЕМ СО СВЯЗЯМИ ВТОРОГО РОДА

В настоящей работе предлагается метод, который сводит Гамильтоновые системы с неприводимыми связями второго рода к системам со связями первого рода.

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1. Introduction

The main goal of this work is to reduce the quantization problem of a Hamiltonian system with second-class constraints [1] in an initial phase space to the quantization problem of a system with first-class constraints in an extended phase space. Such a problem was considered in the work [2] too, but we bring a more simple, to our opinion, solution of this problem.

After such a reduction one can apply the Batalin-Fradkin-Vilkovisky (BFV) [3,4] quantization of the systems with first-class constraints to construct a BRST operator and quantum Hamiltonian for the initial system.

The theory of a system with second-class constraints is formulated in a phase space of canonical variables q^i, p_i $i = 1, \ldots, n$, in terms of which a Hamiltonian $H_0(q^i, p_i)$ is given.

Canonical variables are subject to the constraint equation

$$G_\alpha(q^i, P_i) = 0\,, \quad \alpha = 1, \ldots, 2m\,, \tag{1.1}$$

The constraints have the following Poisson brackets:

$$\{G_\alpha, G_\beta\} = \omega_{\alpha\beta} \tag{1.2}$$

with nonzero determinant on the surface of constraints:

$$\det \omega \Big|_{G_\alpha = 0} \neq 0\,. \tag{1.3}$$

Such a theory has $2n - 2m$ independent (physical) degrees of freedom.

2. The Construction of an Equivalent System with First-Class Constraints

Let us enlarge the phase space by the introduction of the variables $\xi_\alpha, \alpha = 1, \ldots, 2m$ with the same statistics as the constraints G_α and with the following Poisson brackets:

$$\{\xi_\alpha, \xi_\beta\} = (g^{-1})_{\alpha\beta}\,. \tag{2.1}$$

One can take the matrix g, for example, as unit $(g_{\alpha\beta} = \delta_{\alpha\beta})$ in the case of Grassman constraints and as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where 1 is the unit $m \times m$ matrix, in Bose case.

Let us consider the dynamical system in the enlarged phase space with the action

$$S = \int [\xi_\alpha g_{\alpha\beta} \dot{\xi}_\beta + P_i \dot{q}^i - H(P_i, q^i)]\, d\tau \tag{2.2}$$

where $H = H_o(P, q) + \sum_\alpha \lambda_\alpha G_\alpha$ and λ_α are determined from the equations

$$\{H, G_\alpha\} = 0, \quad \alpha = 1, \ldots, 2m. \tag{2.3}$$

Eqs.(2.3) give the following recursion formula for λ_α as a power-series over G :

$$\lambda_\alpha = \sum_{K=0}^{\infty} \lambda_\alpha^{(K)},$$
$$\lambda_\alpha^{(0)} = -\{H_0, G_\beta\}\omega_{\beta\alpha}^{-1}, \tag{2.4}$$
$$\lambda_\alpha^{(K)} = -(-1)^{n(G_\beta)n(G_\delta)}\{\lambda_\delta^{(K-1)} G_\beta\} G_\delta\, \omega_{\beta\alpha}^{-1},$$

where $n(G)$ is the Grassman parity of the constraint G .

If the initial Hamiltonian commutes with constraints $\{H_o, G_\alpha\} = 0$, then $H = H_o$.

Note that from Eqs.(2.3), (1.2) and the Jacobi identity the following equation comes out:

$$\{H, \omega_{\alpha\beta}\} = 0. \tag{2.5}$$

The theory (2.2) has a gauge invariance generated by just $2m$ first-class constraints $\overline{G}_\alpha(P, q, \xi)$:

$$\{\overline{G}_\alpha, \overline{G}_\beta\} = 0 \tag{2.6}$$

$$\{H, \overline{G}_\alpha\} = 0 \tag{2.7}$$

Generally, the new constraints $\overline{G}$ are polynomial over ξ , but in the case

$$\{G_\alpha, \omega_{\beta\gamma}\} = 0 \qquad \alpha, \beta, \gamma = 1, \ldots, 2m \tag{2.8}$$

they are linear:

$$\bar{G}_\alpha = G_\alpha + X^{(1)}_{\alpha\beta}\xi_\beta . \tag{2.9}$$

The matrix $X^{(1)}$ is determined by the equation:

$$X^{(1)}_{\alpha\alpha'} X^{(1)}_{\beta\beta'} g^{-1}_{\alpha'\beta'} = -\omega_{\alpha\beta} \quad (X^{(1)} g^{-1} X^{T(1)} = -\omega) \tag{2.10}$$

which follows from Eq.(2.7).

In the case of Grassman constraints Eq. (2.10) has a solution:

$$X^{(1)} = (-\omega)^{1/2} \tag{2.11}$$

for the Bose constraints:

$$X^{(1)} = g^{-1}(g\omega)^{1/2}. \tag{2.12}$$

In the general case new constraints $\bar{G}$ are polynomial over ξ :

$$\bar{G}_\alpha = G_\alpha + \sum_K X^{(K)}_{\alpha\,\beta_1\cdots\beta_K}\xi_{\beta_1}\cdots\xi_{\beta_K} \tag{2.13}$$

Eqs. (2.7) give Eq.(2.10) in zero order of ξ and the following equations of order ξ^K, $K \geq 1$

$$(K+1)X^{(K+1)}_{\alpha\sigma\gamma_1\cdots\gamma_K} g^{-1}_{\sigma\rho} X^{(1)}_{\beta\rho} + (K+1)X^{(1)}_{\alpha\sigma} g^{-1}_{\sigma\rho} X^{(K+1)}_{\beta\rho\gamma_1\cdots\gamma_K} + \mathcal{D}_{\alpha\beta\gamma_1\cdots\gamma_K} + \tag{2.14}$$

$$+ R_{\alpha\beta\gamma_1\cdots\gamma_K} = 0,$$

where

$$R_{\alpha\beta\gamma_1\cdots\gamma_K} = \left\{ X^{(K-1)}_{\alpha(\gamma_1\cdots\gamma_{K-1}}, X^{(1)}_{\beta\gamma_K)} \right\} + \left\{ X^{(1)}_{\alpha(\gamma_1}, X^{(K-1)}_{\beta\gamma_2\cdots\gamma_K)} \right\} \tag{2.15}$$

and

$$\mathcal{D}_{\alpha\beta\gamma_1\cdots\gamma_K} = \left\{ G_\alpha, X^{(K)}_{\beta\gamma_1\cdots\gamma_K} \right\} + \left\{ X^{(K)}_{\alpha\gamma_1\cdots\gamma_K}, G_\beta \right\} +$$

$$+ \sum_{\ell=2}^{K-2} \left\{ X^{(K-\ell)}_{\alpha(\gamma_1\cdots\gamma_{K-\ell}}, X^{(\ell)}_{\beta\gamma_{K-\ell+1}\cdots\gamma_K)} \right\} + \sum_{\ell=1}^{K-1} (K-\ell+1)(\ell+1) X^{(K-\ell+1)}_{\alpha\sigma(\gamma_1\cdots\gamma_{K-\ell}} X_{\beta\rho\gamma_{K-\ell+1}\cdots\gamma_K)} g^{-1}_{\sigma\rho} \tag{2.16}$$

where $(\cdots\gamma\cdots)$ means (anti)symmetrization over γ_i in the case of (Grassman) Bose constraints.

From Eqs.(2.15) $X^{(K+1)}$ is determined as a power-series over $X^{(1)}$:

$$X^{(K+1)}_{\alpha\gamma\gamma_1\cdots\gamma_K} = X^{(1)}_{\alpha(\gamma} \, y^{(K)}_{\gamma_1\cdots\gamma_K)}, \tag{2.17}$$

$$y^{(K)}_{\gamma_1\cdots\gamma_K} = \sum_{\nu=0}^{\infty} y^{(K)(\nu)}_{\gamma_1\cdots\gamma_K} \tag{2.18}$$

and

$$y^{(K)(0)}_{\gamma_1\cdots\gamma_K} = \frac{1}{2(K+1)\cdot 2m} \, \omega^{-1}_{\alpha\beta} \, \mathcal{D}_{\alpha\beta\gamma_1\cdots\gamma_K}, \tag{2.19}$$

$$y^{(K)(1)}_{\gamma_1\cdots\gamma_K} = \frac{1}{2(K+1)\cdot 2m} \, \omega^{-1}_{\alpha\beta} \, R_{\alpha\beta\gamma_1\cdots\gamma_K}, \tag{2.20}$$

$$y^{(K)(\nu)}_{\gamma_1\cdots\gamma_K} = \frac{1}{2\cdot 2m} \, \omega^{-1}_{\alpha\beta} \Big[X^{(1)}_{\alpha(\gamma_1} \, y^{(K)(\nu-2)}_{\gamma_2\cdots\gamma_K)\sigma} \, g^{-1}_{\sigma\rho} \, X^{(1)}_{\beta\rho} + \tag{2.21}$$

$$+ X^{(1)}_{\alpha\sigma} \, g^{-1}_{\sigma\rho} \, X^{(1)}_{\beta(\gamma_1} \, y^{(K)(\nu-2)}_{\gamma_2\cdots\gamma_K)\rho} \Big] .$$

The new system, just like the initial one, has $2n+2m-2(2m)=2n-2m$ physical degrees of freedom.

It isn't difficult to see that in the gauge $\xi_\alpha=0$ $\alpha=1\ldots 2m$ the constraints $\bar{G}$ transform into G and the action (2.2) into the initial action because $H=H_0$ on the surface of constraints. This is a classical equivalency of the two systems.

Let us pass to quantum equivalency (equivalency of quantum S-matrix).

3. Quantum Equivalency

One can apply BFV [3,4] or Faddeev-Popov procedure to quantize the system (2.2) with the first-class constraints (2.13). The Faddeev-Popov quantization procedure is more suitable to prove the quantum equivalency of the system (2.2) with the initial one having second-class constraints.

The gauge-invariant action, corresponding to the theory with the first-class constraints $\bar{G}_\alpha$, is:

$$S_{GI}=\int[\xi_\alpha g_{\alpha\beta}\dot{\xi}_\beta+P_i\dot{q}^i-H-\lambda^\alpha\bar{G}_\alpha]\,d\tau . \tag{3.1}$$

The gauge symmetry is generated by the generators $\bar{G}_\alpha$. The gauge transformation of ξ_α looks as follows:

$$\delta\xi_\alpha=\varepsilon^\beta\{\bar{G}_\beta\,\xi_\alpha\}=\varepsilon^\beta X^{(1)}_{\beta\gamma}\,g^{-1}_{\gamma\alpha}+\ldots \tag{3.2}$$

The omitted terms in Eq.(3.2) turn into zero on the surface $\xi_\alpha=0$.

Eq.(3.2) shows that we can take $\xi_\alpha = 0$ as the gauge fixing conditions. The quantum theory in this gauge is described by the following functional integral:

$$Z = \int dq\, dp\, d\lambda\, d\xi\, \delta(\xi_\alpha)\, \Delta_{FP} \exp\left[i S_{GI} \right], \tag{3.3}$$

The Faddeev-Popov determinant Δ_{FP} is determined by Eq.(3.2):

$$\Delta_{FP} = S\det X g^{-1} = S\det \omega^{1/2}. \tag{3.4}$$

The integration over λ, ξ in Eq.(3.3) gives

$$Z = \int dq\, dp \left(\prod_\alpha \delta(G_\alpha)\right) S\det{}^{1/2}\{G_\alpha G_\beta\} \exp\left[i\int (p_i \dot{q}^i - H_0)\, d\tau\right] \tag{3.5}$$

which coincides with the well-known expression of the quantum generating functional of the theory with second-class constraints.

Thus, the initial theory with second-class constraints can be replaced by the theory (2.2) with the first-class constraints $\bar{G}_\alpha$. Then applying the BFV quantization procedure the following BRST operator (Ω) and quantum Hamiltonian (H_Ψ) for the initial theory we obtain:

$$\Omega = \bar{G}_\alpha C^\alpha \tag{3.6}$$

and

$$H_\Psi = H + \{\Omega, \Psi\}, \tag{3.7}$$

where C^α are the ghost functions of the statistics opposite to that of functions G_α, and Ψ is an arbitrary function over the enlarged phase space and ghost variables.

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(на английском языке, перевод З.Н.Асланян)

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