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ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ
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CASIMIR OPERATORS FOR QUANTUM sl_n GROUPS

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sl_n ՔՎԱՆՏԱՅԻՆ ԽՄԲԵՐԻ ԿԱԶԻՄԻՐԻ ՕՊԵՐԱՏՈՐՆԵՐԸ

Կառուցված են sl_n քվանտային խմբի Կազիմիրի երկու օպերատոր-ներ: Նրանք կապված են Դինկինի դիագրամի ինքնաձևությամբ, և երկուսն էլ ձգտում են sl_n հանրահաշվի Կազիմիրի սովորական քառակուսային օպերատորին, երբ ձևափոխության պարամետրը ձգտում է մեկի:

Երևանի Ֆիզիկայի ինստիտուտ

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The great interest arose recently to the definite deformation of the enveloping algebras of classic Lie algebras or corresponding affine algebras - the so-called quantum groups [1,2,3]. Their theory is being developed in many directions, in great parallel with the classic case - in particular, works [4,5] develop the vertex operator representation for quantum groups - and show its relevance to many different branches of physics and mathematics, e.g. topological gauge theory [6] , representation theory [7] , knots theory and many others [8].

The purpose of this letter is to find a Casimir operator for quantum sl_n groups, which is a generalization of quadratic Casimir operator for classic groups, i.e. it tends to the standard quadratic Casimir sl_n Lie algebra in the limit, when the parameter q of deformation tends to[1]. We find two such operators and they are connected through the automorphism of Dynkin diagram. The importance of Casimir operators in applications is well known, in particular, it will be necessary in the construction of corresponding deformation of conformal theories with Kac-Moody symmetries [9].

In what follows we define the deformation of the envelopin algebra of sl_n, then define some elements of this algebra and prove that they commute with all elements of algebra.

The quantum group is defined to be the algebra with generators $e_j^{\pm}$, $K_i^{\pm 1}$ and relations [1-3] :

$$K_i e_j^{\pm} = e_j^{\pm} K_i q^{\pm a_{ij}} \tag{1a}$$

$$[e_i^{+}, e_j^{-}] = \delta_{ij}(K_i - K_i^{-1})/(q-q^{-1}) \tag{1b}$$

$$[e_i^{\pm}, e_j^{\pm}] = 0 \qquad a_{ij} = 0 \tag{1c}$$

$$[K_i , K_j] = 0 \tag{1d}$$

$$e_i^{\pm} e_i^{\pm} e_j^{\pm} - (q+q^{-1}) e_i^{\pm} e_j^{\pm} e_i^{\pm} + e_j^{\pm} e_i^{\pm} e_i^{\pm} = 0 \qquad a_{ij} = -1 \tag{1e}$$

a_{ij} is the Cartan matrix for some simple Lie algebra (sl_n in our case).

Let's introduce the elements of algebra, corresponding to all non-simple roots:

$$e_{ij}^{\pm} = [e_{ij-1}^{\pm}, e_j^{\pm}]_q = q^{(1/2)} e_{ij-1}^{\pm} e_j^{\pm} - q^{-(1/2)} e_j^{\pm} e_{ij-1}^{\pm} . \qquad i<j$$

$$e_{ij}^{\pm} = [e_{ij+1}^{\pm}, e_j^{\pm}]_q = q^{(1/2)} e_{ij+1}^{\pm} e_j^{\pm} - q^{-(1/2)} e_j^{\pm} e_{ij+1}^{\pm} . \qquad i>j$$

$$e_{ii}^{\pm} = e_i^{\pm} .$$

So, to each non-simple root there correspond two elements: e_{ij} and e_{ji} .

We define q-commutator between two elements e_{ij}^{α} , $e_{k\ell}^{\beta}$ ($\alpha , \beta = \pm$) as

$$[e_{ij}^{\alpha}, e_{k\ell}^{\beta}]_q = q^{-c} e_{ij}^{\alpha} e_{k\ell}^{\beta} - q^{c} e_{k\ell}^{\beta} e_{ij}^{\alpha} , \tag{2}$$

where c is equal to

, $c = \alpha\beta (A_{ij} , A_{k\ell})/2$

A_{ij} is the sum of all simple roots from i to j, i.e. A_{ij} is the set of all positive roots. (The normalization of scalar product is such that the length of all roots is $\sqrt{2}$.) We shall need some relations between e_{ij}^{α} :

$$[e_{ij}^{\alpha}, e_{j+1}^{\alpha}]_q = e_{ij+1}^{\alpha} , \qquad i \leq j , \tag{3}$$

$$[e_{ij}^{\alpha}, e_{j+1k}^{\alpha}]_q = e_{ik}^{\alpha} , \qquad i \leq j < k , \tag{4}$$

$$[e_{ij}^{\alpha}, e_{j}^{\alpha}]_q = 0 , \qquad i \leq j . \tag{5}$$

$$[e_{ij}^{\alpha}, e_{k}^{\beta}]_q = [e_{ij}^{\alpha}, e_{k}^{\beta}] = 0 , \quad i<k<j , \tag{6}$$

$$[e_{ik}^{-}, e_i^{+}] = -q^{-(1/2)} K_i e_{i-1k}^{-} , \quad k \leq i-1 . \tag{7}$$

One also can add to the relations (3)-(7) relations (3')-(7') which are obtained from (3)-(7) through automorphism of algebra considered, which follows from the automorphism of Dynkin diagram:

$$K_i \to K_{n-i+1}\,, \quad e_i^{\pm} \to e_{n-i+1}^{\pm}\,. \tag{8}$$

Now we shall prove that the following operators C, C' commute with all elements of algebra:

$$C = K\Big[\sum_{i<j} K_1^{-1}\dots K_{i-1}^{-1} K_{j+1}\dots K_n e_{ij}^{+} e_{ji}^{-} q^{i+j-n-1} +$$

$$+\sum_{j=0}^{n} K_1^{-1}\dots K_j^{-1} K_{j+1}\dots K_n q^{2j-n}(q-q^{-1})^{-2}\Big]$$

$$K = K_1^{a_1} K_2^{a_2}\dots K_n^{a_n}\,,$$

where $a_0, a_1, \dots a_n, a_{n+1}$ are the elements of the arithmetic progression with $a_0=1$, $a_{n+1}=-1$:

$$a_i = (n+1-2i)/(n+1)$$

The second operator C' may be obtained from the expression for C through the automorphism (8).

Before beginning the proof one more remark is necessary. Strictly speaking, C, as it stands, is not an element of algebra, due to the non-integer powers of K_i. Different solutions of this problem are possible. One may, for example, introduce logarithmic generators $\ln(K_i)$. But it is interesting that one may take the $(n+1)$-th power of C and all non-integer powers disappear.

The following is the proof of the commutativity of C, C' with all the elements of algebra. It is enough to consider C.

It is evident from (1a) that

$$[C, K_i] = 0$$

Let's consider commutator of C with e_i, $1<i<n$. e_i commutes with K:

$$[K, e_i] = 0$$

Further, e_i commutes with the terms in C containing $e_{k\ell}^{+}$, with $k<i<\ell$ or $i<k$, or $\ell<i$. This follows from (6) and (1c). It remains to consider the commutators with terms containing $e_{k\ell}^{+}$, $i=k-1$, $i=k$, or $i=\ell$, or $i=\ell+1$. We shall show now that the terms with $i=\ell$ cancel with $i=\ell+1$, and terms with $i=k-1$ cancel with terms with $i=k$. It is sufficient to consider the first case, the second one is similar.

We consider at first the commutator of e_i^{+} with term containing $e_{k\,i-1}^{+}$, $k \leq i-1$:

$$[K K_1^{-1}\dots K_{k-1}^{-1} K_i \dots K_n e_{k\,i-1}^{+} e_{i-1\,k}^{-} q^{i+k-1}, e_i^{+}] =$$

$$= K K_1^{-1}\dots K_{k-1}^{-1} K_i \dots K_n e_{k\,i-1}^{+} e_i^{+} e_{i-1\,k}^{-} q^{i+k-1} - \tag{9}$$

$$- K K_1^{-1}\dots K_{k-1}^{-1} K_i \dots K_n e_i^{+} e_{k\,i-1}^{+} e_{i-1\,k}^{-} q^{i+k-2} =$$

$$= K K_1^{-1} \dots K_{k-1}^{-1} K_i \dots K_n [e^+_{ki-1}, e^+_i]_q \, e^-_{i-1k} \, q^{i+k-(3/2)} =$$

$$= K K_1^{-1} \dots K_{k-1}^{-1} K_i \dots K_n e^+_{ki} e^-_{i-1k} \, q^{i+k-(3/2)} =$$

$$= K_i \, (K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^-_{i-1k} \, q^{i+k-(3/2)}) .$$

Then we consider the next commutator $(k \leq i-1)$:

$$[K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^-_{ik} q^{i+k}, e^+_i] =$$

$$K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^-_{ik} q^{i+k} e^+_i -$$

$$- K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_i e^+_{ki} e^-_{ik} q^{i+k+1} =$$

Now we use (7):

$$= K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^+_i e^-_{ik} q^{i+k} -$$

$$- q^{-(1/2)} K_i \, (K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^-_{i-1k} q^{i+k-1}) -$$

$$- K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_i e^+_{ki} e^-_{ik} q^{i+k+1}$$

Using (5), we have:

$$= -q^{-(1/2)} K_i (K K_1^{-1} \dots K_{k-1}^{-1} K_{i+1} \dots K_n e^+_{ki} e^-_{i-1k} \, q^{i+k-1}) . \qquad (10)$$

We see that the sum of (9) and (10) is equal to zero. In a similar way one can show the cancellation of results of commu-

tation of e^+_i with terms containing e^+_{ik}, and with terms containing e^+_{i+1k} $k \geq i+1$.

It remains to consider the commutator of e^+_i with terms containing e^+_i, and without e^+_k:

$$[K K_1^{-1} \dots K_{i-1}^{-1} K_{i+1} \dots K_n e^+_i e^-_i q^{2i}, e^+_i] =$$

$$= K K_1^{-1} \dots K_{i-1}^{-1} K_{i+1} \dots K_n e^+_i q^{2i} (K_i^{-1} - K_i)/(q - q^{-1}) =$$

$$= K K_1^{-1} \dots K_{i-1}^{-1} K_{i+1} \dots K_n q^{2i} (q^2 K_i^{-1} - q^{-2} K_i) e^+_i / (q - q^{-1}) \qquad (11)$$

$$[K \sum_{j=0}^{n} K_1^{-1} \dots K_j^{-1} K_{j+1} \dots K_n q^{2j+1} (q - q^{-1})^{-2}, e^+_i] =$$

$$= K K_1^{-1} \dots K_{i-1}^{-1} K_{i+1} \dots K_n (q-q^{-1})^{-2} ((1-q^2) q^{2i+1} K_i^{-1} + (1-q^{-2}) q^{2i-1} K_i) e^+_i =$$

$$= K K_1^{-1} \dots K_{i-1}^{-1} K_{i+1} \dots K_n (q-q^{-1})^{-1} (-q^{2i+2} K_i^{-1} + q^{2i-2} K_i) e^+_i \qquad (12)$$

The sum of (11) and (12) is zero. In the similar way one can prove the commutativity of e^+_i with terms in C containing e^+_{kl}, $i \leq k$. It remains to prove the commutativity of C with e_1 and e_n. It is enough to consider e_n, the proof for e_1 is similar.

The difference from the previous case is that now there is no K_{n+1}, and its role exactly is played by K:

$$K e^{\pm}_n = e^{\pm}_n K q^{\mp 1} \qquad (\text{compare } K_{n+1} e^{\pm}_n = e^{\pm}_n K_{n+1} q^{\mp 1})$$

In all other respects the proof proceeds as in the case of e_i , $1 < i < n$, and this finishes the proof of the whole statement.

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