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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE

D.R. KARAKHANYAN

# INDUCED DIRAC OPERATOR AND SMOOTH MANIFOLD GEOMETRY

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D.R. KARAKHANYAN

# INDUCED DIRAC OPERATOR AND SMOOTH MANIFOLD GEOMETRY

The determinant of the Dirac operator induced from arbitrary smooth manifold on Riemann surface is calculated explicitly.

Yerevan Physics Institute

Yerevan 1990

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Д.Р.КАРАХАНЯН

# ИНДУЦИРОВАННЫЙ ОПЕРАТОР ДИРАКА И ГЕОМЕТРИЯ ГЛАДКИХ МНОГООБРАЗИЙ

Вычислен детерминант индуцированного из произвольного гладкого многообразия на двухмерную поверхность оператора Дирака.

Ереванский физический институт
Ереван 1990

The induced Dirac operator emerged for the first time in Ref. [1] as a naive continuous limit of the three-dimensional Ising model. The determinant of the Dirac operator induced on surface $\Sigma$ by spinor structure of $\mathbb{R}^3$ wherein $\Sigma$ is embedded was calculated in Ref. [2] . This result was generalized for the $\mathbb{R}^d$ case in Ref. [3] .

Using the structure similarity between this theory and the Green-Schwarz superstring theory it turned out possible to apply the bosonization technique developed in Ref. [2] to the heterotic superstring theory [4] .

In the present paper the result of [3] is generalized for the case of arbitrary smooth manifold $M$ .

Let $M$ be arbitrary smooth manifold wherein the two-dimensional surface $\Sigma$ is embedded.

We shall be interested only in local properties of this embedding and not in global restrictions on it. Therefore we shall work in the limits of one coordinate vicinity on $\Sigma$ with the coordinates $\xi^{\alpha}$ ( $\alpha$ = 1,2) and one coordinate vicinity on $M$ with the coordinates $X^{\mu}$ ( $\mu$ = 1,...,d).

By means of the tetrads $E^{\mu}_{m}(X)$ :

$$E_{\mu}^{m}(X)\cdot E_{\nu}^{m}(X)=G_{\mu\nu}(X) \tag{1}$$

we can determine Dirac action in $M$ :

$$S_D=\frac{i}{2}\int_M d^dX\cdot\sqrt{G}\cdot E^{\mu}_{m}(\bar{\psi}\gamma^m\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma^m\psi) \tag{2}$$

which possesses local SO(d)-invariance.

Here $G=\det\|G_{\mu\nu}\|$ ; $\psi(X)$ are the spinors of the space tangential to $M$ (SO(d) representation).

Let the surface $\Sigma$ be described in $M$ by the equation:

$$X^{\mu}=X^{\mu}(\xi^{\alpha}) \tag{3}$$

Then the projection $P:M\rightarrow\Sigma$ can be realized using the projective operator:

$$P^{\mu\nu}=\partial_\alpha X^{\mu}\cdot g^{\alpha\beta}\cdot\partial_\beta X^{\nu} \tag{4}$$

where $g^{\alpha\beta}$ is the metric induced from $M$ on $\Sigma$ :

$$g^{\alpha\gamma}\cdot g_{\gamma\beta}=\delta^{\alpha}_{\beta}\quad,\qquad g_{\alpha\beta}=\partial_\alpha X^{\mu}\cdot G_{\mu\nu}\cdot\partial_\beta X^{\nu} \tag{5}$$

The two-dimensional field theory of interest is obtained from (2) using projector (4):

$$S_{2D}=\frac{i}{2}\int_\Sigma d^2\xi\sqrt{g}\cdot g^{\alpha\beta}\cdot\partial_\alpha X^{\mu}\cdot E_{\mu}^{m}(\bar{\psi}\gamma_m\partial_\beta\psi-\partial_\beta\bar{\psi}\gamma_m\psi) \tag{6}$$

Now we introduce the vectors tangential $\partial_\alpha X^{\mu}$ and normal $n^{\mu}_{i}$ ( $i$ = 1,...,d-2) to the surface:

$$\partial_\alpha X^\mu \cdot G_{\mu\nu} \cdot n_i^\nu = 0 \ , \quad n_i^\mu \cdot G_{\mu\nu} \cdot n_j^\nu = \delta_{ij} \ , \quad i,j = 1,..d-2 \qquad (7)$$

Orthonormalizing the tangential vectors using two-dimensional tetrads

$$X_a^\mu(\xi) = e_a^\alpha(\xi) \cdot \partial_\alpha X^\mu(\xi) \ , \quad e_a^\alpha(\xi) \cdot e_a^\beta(\xi) = g^{\alpha\beta} \qquad (8)$$

we obtain orthonormalized basis in each point of the surface:

$$(X_a^\mu, n_i^\mu) \qquad (9)$$

With the use of local SO(d)-rotation the basis (9) in every point of $\Sigma$ can be reduced to some fixed form ( $\overset{(0)}{X}{}_a^\mu$ , $\overset{(0)}{n}{}_i^\mu$ ) independent of $\xi^\alpha$ .

Or in the spinor representation:

$$\begin{aligned} \gamma_a(\xi) &= X_a^m(\xi) \cdot \gamma_m = \Omega^{-1}(\xi)\sigma_a \Omega(\xi) \\ \gamma_i(\xi) &= n_i^m(\xi) \cdot \gamma_m = \Omega^{-1}(\xi)\sigma_i \Omega(\xi) \end{aligned} \qquad (10)$$

where $X_a^m = X_a^\mu \cdot E_\mu^m$

$n_i^m = n_i^\mu \cdot E_\mu^m$

and ( $\sigma_a$ , $\sigma_i$ ) is some fixed basis of d-dimensional Dirac $\gamma$-matrices which differs from $\gamma_m$ in no more than global rotation.

From (10) we can readily obtain:

$$\Omega^{-1}d\Omega = \frac{1}{4}(\gamma_a\, d\gamma^a + \gamma_i\, d\gamma^i) = \tag{11}$$
$$= \frac{1}{4}\Gamma^{ab}\gamma_{ab} + \frac{1}{2}h^{ai}\gamma_{ai} + \frac{1}{4}H^{ij}\gamma_{ij}$$

where $\Gamma$, $h$ and $H$ fields are defined as expansion coefficients of $d\gamma^a$ and $d\gamma^i$ over basis (10):

$$\begin{aligned} d\gamma^a &= \Gamma^{ab}\gamma_b + h^{ai}\gamma_i \\ d\gamma^i &= -h^{ai}\gamma_a + H^{ij}\gamma_j \end{aligned} \tag{12}$$

The integrability conditions of system (12) provide the condition of embedding of $\Sigma$ in $M$ in terms of $\Gamma$, $h$ and $H$ fields:

$$\begin{aligned} d\Gamma^{ab} &= -h^{ai}\cdot h^{bi} \\ dh^{ai} &= -\Gamma^{ab}h^{bi} - H^{ij}h^{aj} \\ dH^{ij} - H^{ik}_{\cdot}H^{kj} &= -h^{ai}_{\cdot}h^{aj} \end{aligned} \tag{13}$$

Differentiating the (5) we can readily obtain a relation:

$$\nabla_\alpha(\partial_\beta X^\mu) = h^i_{\alpha\beta}\cdot n_i{}^\mu \tag{14}$$

which defines a symmetric second quadratic form of the surface $h^i_{\alpha\beta}$. Here the covariant derivative affects both curved indices $\alpha$ and $\mu$.

The fields defined by expansion (12)

$$\Gamma_\alpha^{ab} = \frac{1}{tr 1} \cdot tr(\gamma^b \partial_\alpha \gamma^a) = e^b_\beta \nabla_\alpha e^{\beta a} - \omega_\alpha^{mn} X^a_m X^b_n$$

$$h_\alpha^{ai} = \frac{1}{tr 1} \cdot tr(\gamma^i \partial_\alpha \gamma^a) = e^{\beta a} \cdot h^i_{\alpha\beta} - \omega_\alpha^{mn} X^a_m \cdot n^i_n \tag{15}$$

$$H_\alpha^{ij} = \frac{1}{tr 1} \cdot tr(\gamma^j \partial_\alpha \gamma^i) = n^j_\mu \cdot \partial_\alpha X^\nu \cdot \nabla_\nu n^{i\mu} - \omega_\alpha^{mn} n^i_m n^j_n$$

differ from the $\overset{(0)}{\Gamma}$, $\overset{(0)}{h}$ and $\overset{(0)}{H}$ fields corresponding to them in the case $M = R^d$ only by one term which contains spinor connection of space $M$, $\omega_\alpha^{mn} = \partial_\alpha X^\nu \cdot E^n_\mu \cdot \nabla_\nu E^{\mu m}$, the presence of which in the action reflects its local SO(d)-symmetry.

The relation (11) can be rewritten in the form:

$$d\Omega \cdot \Omega^{-1} = d\overset{(0)}{\Omega} \cdot \overset{(0)}{\Omega}{}^{-1} - \frac{1}{4} \omega^{mn} \cdot \Omega \gamma_{mn} \Omega^{-1} \tag{16}$$

which makes the relationship between the case $M = R^d$ and the general case explicit.

The induced Dirac operator corresponding to the action (6):

$$D = i e^\alpha_a \Omega^{-1} \sigma^a \left(\partial_\alpha - \frac{1}{4} \overset{(0)}{\Gamma}{}_\alpha^{bc} \cdot \sigma_{bc} - \frac{1}{4} H_\alpha^{ij} \sigma_{ij}\right) \Omega + \frac{i}{8} e^\alpha_a \omega_\alpha^{mn} \{\gamma^a; \gamma^{mn}\} \tag{17}$$

The last term due to which $D$ differs from the Dirac operator considered in Ref. [3] does not allow to write the action (6) in the form of two-dimensional field theory of fermions interacting with the gravitation and gauge field.

From (17) we can see that along with the reparametrizational and local SO(d) invariances the action also possesses the local SO(2) x SO(d-2) invariance acquired just as in the $M=R^d$ case owing to the transition from $X^\mu$ to $\Gamma$, $h$ and $H$ fields.

Indeed, in Ref. [3] we had the action $\frac{i}{2}\int\limits_\Sigma d\xi\sqrt{g}\,g^{\alpha\beta}\partial_\alpha X^\mu\cdot(\bar\psi\gamma_\mu\partial_\beta\psi - \partial_\beta\bar\psi\gamma_\mu\psi)$ where $g_{\alpha\beta}=\partial_\alpha X^\mu\cdot\partial_\beta X^\mu$, i.e. d independent bosonic degrees of freedom combined in a reparametrization-invariant way. To integrate over $\psi$ we reduced it to the form $\int\limits_\Sigma d\xi\sqrt{g}\,\bar\Phi\hat{D}^{(\varphi)}\psi$ where $\hat{D}^{(\varphi)}=i e^{\alpha a}\Omega^{-1}(\partial_\alpha-\frac{1}{4}\Gamma^{bc}_\alpha\sigma_{bc}-\frac{1}{4}H_\alpha^{ij}\sigma_{ij})\Omega$. The d-2 bosonic degrees of freedom are distributed between $e_\alpha^a$, $\Gamma_\alpha^{ab}$, $h_\alpha^{ai}$ and $H_\alpha^{ij}$ fields with respect to symmetricity of $h^i_{\alpha\beta}=h^i_{\beta\alpha}$ and integrability conditions as well as the action symmetries in these variables as follows:

$$4+2+3(d-2)+(d-2)(d-3)-\frac{d(d-1)}{2}-1-\frac{(d-2)(d-3)}{2}-2-2=d-2$$

The similar calculation in the case of arbitrary manifold for the action (6) gives:

$$d^2+d-\frac{d(d-1)}{2}-2$$

Similarly d quantities $X^\mu$ are replaced by

$$4+3(d-2)+(d-2)(d-3)-\frac{d(d-1)}{2}$$

of quantities $e_\alpha^a$, $h^i_{\alpha\beta}$, ($\Gamma_\alpha^{ab}=e_\beta^b\nabla_\alpha e^{\beta a}$), $H_\alpha^{ij}$ constrained by $\frac{1}{2}d(d-1)$ integrability conditions and local SO(2) x SO(d-2) symmetry.

If now we substitute in (17) the expansion

$$\gamma_m = X_m^a\cdot\gamma_a + n_m^i\cdot\gamma_i \tag{18}$$

and use a $\frac{SO(d)}{SO(2)\times SO(d-2)}$ coset-part of local SO(d)-symmetry for imposition of gauge condition, we'll have:

$$e^{\alpha a}\omega_\alpha{}^{mn}X^b_m\cdot n^i_n = e^{\alpha b}\omega_\alpha{}^{mn}X^a_m n^i_n \tag{19}$$

This is possible because at SO(d) rotations

$$\gamma_a \to W^{-1}(\xi)\gamma_a W(\xi)\ ,\ \gamma_i \to W^{-1}(\xi)\gamma_i W(\xi)\ ,\ \Omega\to\Omega(\xi)W(\xi)$$

i.e.

$$\Omega^{-1}d\Omega \to W^{-1}(\Omega^{-1}d\Omega + dW\cdot W^{-1})W$$

and choosing $dW\cdot W^{-1} = W^{ai}\cdot\frac{1}{2}\gamma_{ai} \in \frac{SO(d)}{SO(2)\times SO(d-2)}$

we can attain that the expression $(h_\alpha{}^{ai} + W_\alpha{}^{ai})e^{\alpha b}$ would be symmetric over $a$ , $b$ , which corresponds to gauge condition (19) if we recall equation (14).

In gauge (19) the action (6) takes a form usual for two-dimensional field theory:

$$S_{1D} = \frac{i}{2}\int_\Sigma d^2\xi\sqrt{g}\,e^\alpha_a\bar{\Psi}\Omega^{-1}\sigma^a(\partial_\alpha - \frac{1}{4}e^c_\beta\nabla_\alpha e^{\beta b}\sigma_{bc} - \frac{1}{4}H_\alpha{}^{ij}\sigma_{ij})\Omega \tag{20}$$

The Dirac operator corresponding to (20) anticommutes with matrix $\gamma_{+-}$ , therefore the left- and right-side fermions in (20) separate from each other.

We parametrize the tetrads in the form:

$$e^a_\alpha\cdot\sigma_a = \rho^{1/2}\cdot\partial_\alpha f^a\cdot\lambda^{-1}\sigma_a\lambda \tag{21}$$

To calculate the actions of the left fermions we fix reparametrizational invariance by imposing conformal gauge:

$$g_{\alpha\beta}(\xi)=\rho(\xi)\cdot\delta_{\alpha\beta} \quad \text{or} \quad \begin{array}{l} \hat{e}^{+}=e^{+}_{\alpha}\cdot\sigma^{a}=\rho^{-1/2}\cdot\lambda^{-1}\sigma^{+}\lambda \\ \hat{e}^{-}=e^{-}_{\alpha}\sigma^{a}=\rho^{-1/2}\cdot\lambda^{-1}\sigma^{-}\lambda \end{array} \tag{22}$$

The Dirac operator in gauge (22):

$$D=i\Omega^{-1}\lambda^{-1}\rho^{-3/4}\left[\sigma_{-}(\partial_{+}+H_{+})+\sigma_{+}(\partial_{-}+H_{-})\right]\rho^{1/4}\lambda\Omega=D_{L}+D_{R} \tag{23}$$

where $H_{\alpha}=-\frac{1}{4}H_{\alpha}{}^{ij}\sigma_{ij}$ .

Now we define variation of determinant of Dirac chiral operator:

$$\begin{aligned} \delta W=\delta\log\det D_{L}&=\lim_{\varepsilon\to 0}F.P.\int_{\varepsilon}^{\infty}ds\,Tr\left(\delta D_{L}\cdot D_{R}\,e^{-sD_{L}D_{R}}\right)= \\ &=\lim_{\varepsilon\to 0}F.P.\,Tr\Big[(\Omega^{-1}\delta\Omega+\Omega^{-1}\lambda^{-1}\delta\lambda\Omega+\Omega^{-1}g^{-1}\delta g\Omega+\tfrac{1}{4}\delta\log\rho)e^{-\varepsilon D_{R}D_{L}}+ \\ &+(-\Omega^{-1}\delta\Omega-\Omega^{-1}\lambda^{-1}\delta\lambda\Omega-\Omega^{-1}g^{-1}\delta g\Omega-\tfrac{3}{4}\delta\log\rho)e^{-\varepsilon D_{L}D_{R}}\Big] \end{aligned} \tag{24}$$

wherein we parametrized

$$H_{+}=g^{-1}\partial_{+}g \tag{25}$$

From here the Ward identities for effective action are

$$\begin{aligned} \delta W=&-\frac{1}{4\pi}\int_{\Sigma}d^{2}\xi\,tr\left(\delta\Omega\cdot\Omega^{-1}(\overset{(0)}{\Gamma}_{+}+H_{+})\right)-\frac{1}{4\pi}\int_{\Sigma}d^{2}\xi\,tr\left(H_{-}g^{-1}\delta g\right) \\ &-\frac{1}{4\pi}\int_{\Sigma}tr\left(\Gamma_{+}\lambda^{-1}\delta\lambda\right)+\frac{tr\,\mathbf{1}}{48\pi}\int_{\Sigma}d^{2}\xi\cdot\partial_{+}\partial_{-}\log\rho\cdot\delta\log\rho \end{aligned} \tag{26}$$

To integrate equation (26) we impose a gauge condition fixing SO(2) x SO(d-2) local invariance:

$$\partial_+ \Gamma_-^{ab} + \partial_- \Gamma_+^{ab} = \partial_+ H_-^{ij} + \partial_- H_+^{ij} = 0 \tag{27}$$

The integrable expression corresponding to the first term in (26) is

$$\frac{1}{24\pi} \delta \int_{\Sigma_t} tr(d\Omega \cdot \Omega^{-1})^3 = -\frac{1}{\ } \int_{\Sigma} tr(\delta\Omega \cdot \Omega^{-1}(\Gamma_{+-} + H_{+-})$$

where $\Sigma_t$ is a three-dimensional region in $M$ with surface $\Sigma$ as a boundary.

Using the integrability conditions of (13) we can readily show that

$$\int_{\Sigma_t} tr(d\Omega \cdot \Omega^{-1})^3 = -\frac{3}{8} tr\,\mathbb{1} \cdot \int_{\Sigma_t} \left(\Gamma^{ab} d\Gamma_{ab} + H^{ij}(dH_{ij} - \frac{2}{3} H_{ik} H_{kj})\right)$$

A final expression for effective action of the left fermions will be

$$\begin{aligned} W = \frac{tr\,\mathbb{1}}{192\pi} \int_{\Sigma} d\xi \Big[ \sqrt{g} (\overset{(0)}{\Gamma_\alpha} - \lambda^{-1}\partial_\alpha \lambda) g^{\alpha\beta} (\overset{(0)}{\Gamma_\beta} - \lambda^{-1}\partial_\beta\lambda) - \\ - 2\epsilon^{\alpha\beta} \overset{(0)}{\Gamma_\alpha} \cdot \lambda^{-1} \partial_\beta \lambda \Big] + \frac{tr\,\mathbb{1}}{96\pi} \int_{\Sigma} \Big[ \Gamma^{ab} \cdot d\Gamma_{ab} + \\ + H^{ij} (dH - \frac{2}{3} H^2)_{ij} \Big] + \frac{1}{8\pi} \int_{\Sigma} d\xi \sqrt{g} g^{\alpha\beta} tr(H_\alpha H_\beta) \\ + \frac{1}{4\pi} \Big[ \frac{1}{2} \int_{\Sigma} d\xi \sqrt{g} g^{\alpha\beta} tr(g^{-1}\partial_\alpha g \cdot g^{-1} \partial_\beta g) - \frac{i}{3} \int_{\Sigma_t} tr(g^{-1} dg)^3 \Big] \end{aligned} \tag{28}$$

The last two lines appear in case $dim\, M > 3$ . The first two lines, according to Ref. [5] interpretation, define a new covariant local action of two-dimensional gravitation.

Except for the first three terms all the rest are modified by the contribution of d-dimensional spinor connection compared to the $M = R^d$ case.

In view of the necessity of compactification of "superfluous" space measurements in superstring theory, the problem of quantization of superstrings propagating in arbitrary smooth manifold is highly urgent.

As far as the theory of induced Dirac action is a simplified model of string theory (see Ref.[4] ), the present work seems to be promising in this direction.

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The address for requests:
Information Department
Yerevan Physics Institute
Alikhanian Brothers 2,
Yrevan, 375036
Armenia, USSR

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