Preprint YERPHI-1260(46)-90

(EFI-1260-46-90)

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE

A.R. KAVALOV, R.L. MKRTCHYAN

THE LATTICE ABELIAN CHERN-SIMONS GAUGE THEORY

ЦНИИатоминформ
ЕРЕВАН-1990

Նախատիպ ԵՖԻ-1260(46)-90

Ա.ԿԱՎԱԼՈՎ, Ռ.Լ.ՄԿՐՏՉՅԱՆ

ՉԵՐՆ-ՍԱՅՄՈՆԻ ՑԱՆՑԱՅԻՆ ԱԲԵԼՅԱՆ ՏՐԱՄԱՉԱՓԱՅԻՆ ՏԵՍՈՒԹՅՈՒՆ

Ուսումնասիրված են նախապես կառուցված Չերն-Սայմոնի ցանցային տրամաչափային աբելյան տեսության մի քանի հատկությունները: Կառուցված է նաև BF-համակարգի ցանցային տարբերակը, որոնց երկու տեսությունների հատկություններ համապատասխանության մեջ են այդ տեսությունների անընդհատ տարբերակների հատկությունների հետ: Բացահայտված է քննարկվող տեսությունների կապը երկչափ ցանցային վիճակագրական համակարգերի հետ, ինչպես նաև ցույց է տրված ցանցային օղակների Ֆըրեյմինգի ներմուծման անհրաժեշտությունը անընդհատ սահմանի գոյության համար:

Երևանի Ֆիզիկայի Ինստիտուտ

Երևան 1990

Preprint YERPHI-1260(46)-90

Al.R. KAVALOV, R.L.MKRTCHYAN

THE LATTICE ABELIAN CHERN-SIMONS GAUGE THEORY

Some properties of the previously proposed lattice version of abelian Chern-Simons gauge theory are studied. The lattice analog of BF-systems is constructed, the properties of both theories are found to be in close correspondence with those of continuous theory. The correspondence with two-dimensional lattice statistical systems is established and the lattice origin of framing of Wilson loops is shown.

Yerevan Physics Institute
Yerevan 1990

Препринт ЕФИ-1260(46)-90

М.Р. КАБАЛОВ, Р.Л. МКРТЧЯН

РЕШЕТОЧНАЯ АБЕЛЕВА КАЛИБРОВОЧНАЯ ТЕОРИЯ ЧЕРНА-САЙМОНА

Исследованы некоторые свойства построенной ранее решеточной калибровочной абелевой теории Черна-Саймона. Построен также решеточный вариант BF-систем, свойства обеих теорий находятся в полном соответствии со свойствами непрерывных вариантов этих теорий. Установлена связь рассматриваемых теорий с двухмерными решеточными статистическими системами, а также показана необходимость введения фрейминга решеточных петель для существования непрерывного предела.

Ереванский физический институт

Ереван 1990

1. Introduction

The aim of the present paper is to study in some details the simplest properties of the previously constructed three-dimensional abelian lattice Chern-Simons gauge theory [1] and to extend the construction to the case of BF-systems. Our strategy will be to try to keep as close as possible to the corresponding continuous situation and the results we shall obtain will have natural continuous analogs.

The gauge theories with the action independent of the metric (hence, topological) were first considered by A.Schwarz [2,3] who has shown that the partition function is determined in terms of the Ray-Singer analytic torsion. The Chern-Simons theory has also been studied (in abelian case) more recently by A.Polyakov [4] in connection with the Fermi-Bose transmutation, and was made very popular by E.Witten [5,6] who has shown its numerous connection with different fields of physics and mathematics. Let's mention that it provides a natural three-dimensional framework for a knot theory [5], a new description of three-dimensional gravity [6], and, most intriguing, is closely related to the two-dimensional conformal theories [5]. These points have been further studied in a number of works.

A related class of models is given by the BF-systems

[2,3,7-9], which have the nice property of being generalizable to higher dimensions without acquiring higher-derivative terms in the action.

There are different motivations for the lattice approach to the topological theories. The first is the observation that the big part of the relevant information about the topology of the manifold is maintained when replacing the manifold by the corresponding simplicial or cell complex. One may then hope to get the simpler description of the results of the continuous approach. The second, like all other lattice theories, our one provides a natural regularization of the continuous theory. The third is that the lattice approach permits one to define a topological gauge theory for discrete abelian groups, e.g. Z_n.

A simple example of the lattice topological theory was given by J.F.Wheater in the work [10] where he considered a simple two-dimensional topological Ising (i.e. with variables taking values ±1) theory on a triangulated manifold. The important point of the approach of Ref. [10] was that the weight exp(-action) was constructed to be independent of the triangulation - recall that in two dimensions summing over all possible triangulations of the manifolds is equivalent to integrating over all possible metrics in the continuum [11-15].

The lattice topological gauge theories in three dimensions were proposed also by Dijkgraaf and Witten [16]. The action in their work is defined on the flat gauge connections (i.e. the holonomy of the gauge field around any elementary loop of the lattice is trivial). In our construction this limitation does not emerge.

The paper is organized as follows. In Sect.2 we construct the action for abelian lattice Chern-Simons theory for different lattices (for a simplicial lattice the construction was described in Ref.[1]) and actions for lattice BF-system. In Sect.3 the quantum averages of Wilson loops are considered. The corresponding skein relation is just the Makeenko-Migdal loop equation (for a review and a list of relevant references see [17]) for corresponding gauge theory. This interpretation is in agreement with the derivation of skein relation in 1/k expansion in continuum [18]. In Sect.4 we establish an exact relation of the lattice theories considered on the manifold with boundary to the two-dimensional statistical models (the continuum analog may be found in Refs.[5,19]

2. The actions

Consider the simplicial complex corresponding to some closed orientable triangulated manifold M_3 (see any textbook in homology theory e.g. [20]). The standard and useful notation for the n-simplex is by writing down its vertexes in some order. The order of vertexes corresponds to the orientation of the simplex, the two simplexes with the order of vertexes connected by the even (odd) permutation having the same (opposite) orientation. The boundary operator ∂ : (n-chains) $\to$ ((n-1)-chains) acts on the simplexes in the following way:

$$\partial\,[\alpha_0\alpha_1\alpha_2\ldots\alpha_n] = [\alpha_1\alpha_2\ldots\alpha_n] - [\alpha_0\alpha_2\ldots\alpha_n] + \ldots + (-1)^n[\alpha_0\alpha_1\ldots\alpha_{n-1}] \qquad (1)$$

and has the property $\partial^2 = 0$. Evidently, the orientation of the simplex induces the orientations of the components of its boundary. The orientations of the two neighbouring n-simplexes are said to be equal (opposite) if they induce the opposite (equal) orientation on their common (n-1)-simplexes. For the orientable manifolds M^n it is possible to choose the same orientation for all n-simplexes. In the present work we will always make such a choice.

The gauge field A is a 1-cochain, i.e. a linear map of 1-simplexes to the ring of coefficients. We take it to be antisymmetric:

$$A([\alpha_0\alpha_1]) = -A([\alpha_1\alpha_0]). \tag{2}$$

The coboundary operator δ is the operator conjugated to ∂, i.e. for arbitrary n-cochain F

$$\delta F([\alpha_0 \ldots \alpha_{n+1}]) = F(\partial\, [\alpha_0 \ldots \alpha_{n+1}]). \tag{3}$$

Evidently, $\delta^2=0$ as follows from $\partial^2=0$. The gauge transformation of gauge field A is defined to be

$$A \to A + \delta\phi \tag{4}$$

for some 0-cochain ϕ. Note that 0-cochains are defined on 0-simplexes i.e. vertexes and eq. (4) means

$$A([\alpha_0\alpha_1]) \to A([\alpha_0\alpha_1]) + \phi([\alpha_1]) - \phi([\alpha_0]) \tag{5}$$

As another example consider the 2-cochain δA :

$$\delta A([\alpha_0\alpha_1\alpha_2]) = A([\alpha_0\alpha_1]) + A([\alpha_1\alpha_2]) + A([\alpha_2\alpha_0]) \tag{6}$$

which is the (abelian) lattice analog for the field strength.

Let's define now the so-called Kolmogorov-Alexander product, or cup-product ($\cup$-product) in the space of cochains. Given the p-cochain P and s-cochain S one can form a p+s cochain $P_\cup S$ by the formula

$$P_\cup S\,([\alpha_0 \ldots \alpha_{p+s}]) = P([\alpha_0 \ldots \alpha_p])S([\alpha_p \ldots \alpha_{p+s}]). \tag{7}$$

The important properties of this product are the associativity and the graded Leibnitz rule with respect to δ:

$$P \cup (Q \cup S) = (P \cup Q) \cup S \qquad (8)$$

$$\delta(P \cup S) = \delta P \cup S + (-1)^p P \cup \delta S \qquad (9)$$

Note that there is no canonical relation between $P \cup S$ and $S \cup P$.

Using the described objects one naturally constructs the following action [1]

$$S = k \sum_{M_3} \sum_{\sigma} (-1)^{\sigma} A \cup \delta A \qquad (10)$$

Here the first sum is over the tetrahedrons forming the triangulation of the manifold M_3 and $\sum_{\sigma} (-1)^{\sigma}$ denotes the weighted sum over the renumerations of the vertexes of the given tetrahedron: the even (odd) permutations enter with the sign + (-), respectively. k is a real coupling constant. It is checked immediately that due to the properties (8), (9) and $\delta^2=0$ the lagrangian in (10) changes by a "total derivative" $\sum_{\sigma}(-1)^{\sigma}\delta(\phi \cup \delta A)$ under a gauge transformations (4). Thus, the action for boundaryless M_3 is gauge invariant.

It's easy to see that the sum over permutations produces the following value for the action on the tetrahedron with vertexes $\alpha_0,\dots,\alpha_3$:

$$4k(A([\alpha_0\alpha_1])A([\alpha_2\alpha_3])+A([\alpha_0\alpha_2])A([\alpha_3\alpha_1])+A([\alpha_0\alpha_3])A([\alpha_1\alpha_2])) \qquad (11)$$

The geometrical picture is simple - action is the sum of products of pairs of non-intersecting links. In the continuum limit and for U(1) gauge group the standard abelian Chern-Simons action is recovered immediately.

The action (10) was proposed in Ref.[1] as the lattice version of abelian Chern-Simons theory. A point we want to stress here is that its construction is very natural and uses very standard notions of (co)homology theory. One may easily write down the analogous expression for arbitrary (not necessarily simplicial) complex. The one very useful in actual calculations is the case of regular cubic lattice. The relevant formulae look as follows.

The boundary operator acting on a standard cube $I^n = \{0 \leq x_i \leq 1,\ i=1,\ldots,n\}$ is now

$$\partial\, I^n = \sum_{i=0}^{n} (-1)^i (\lambda_i^1 I^n - \lambda_i^c I^n) \qquad (12)$$

where $\lambda_i^{\varepsilon} I^n$ is the face $x_i=\varepsilon$, $\varepsilon=0,1$, of this cube, identified in a natural way with the standard cube I^{n-1} of dimensionality n-1.

The cup product of two cochains P and S of dimensionality p and s respectively is defined by

$$P \cup S\ (I) = P(\lambda_1^0 \ldots \lambda_s^0 I^{p+s})\ S(\lambda_{s+1}^1 \ldots \lambda_{p+s}^1 I^{p+s}) \qquad (13)$$

$$- P(\lambda_1^0 \ldots \lambda_{s-1}^0 \lambda_{s+1}^0 I^{p+s})\ S(\lambda_s^1 \lambda_{s+2}^1 \ldots \lambda_{p+s}^1 I^{p+s}) + \ldots$$

and for small values of p, s is equal to :

p=0, s arbitrary

$$P \cup S\ (I^s) = P(\lambda_1^0 \ldots \lambda_s^0 I^s)\ S(I^s)$$

p=s=1

$$P \cup S\ (I^2) = P(\lambda_1^0 I^2)\ S(\lambda_2^1 I^2) - P(\lambda_2^0 I^2)\ S(\lambda_1^1 I^2)$$

p=1, s=2 (14)

$$P_\cup S\,(I^3) = P(\lambda_1^0\lambda_2^0 I^3)\; S(\lambda_3^1 I^3) - P(\lambda_1^0\lambda_3^0 I^3)\; S(\lambda_2^1 I^3) + $$
$$+ P(\lambda_2^0\lambda_3^0 I^3)\; S(\lambda_1^1 I^3)$$

The properties (8) and (9) remain valid. In cubic case it is not necessary to perform a sum over permutations. The gauge invariant action is given by the sum over the cubes of lagrangian, similar to (10)

$$S = k \sum_{M_3} A_\cup \delta A = = A(\lambda_1^0\lambda_2^0 I^3)\delta A(\lambda_3^1 I^3) - A(\lambda_1^0\lambda_3^0 I^3)\delta A(\lambda_2^1 I^3) \tag{15}$$
$$+ A(\lambda_2^0\lambda_3^0 I^3)\delta A(\lambda_1^1 I^3)$$

Continuous limit may be checked immediately.

Now we turn to the actions for BF-systems, which, as will be shown later, can also be used for the calculation of answers in Chern-Simons theory and are even preferable for obtaining the smooth continuum limit.

The action for BF-systems may be obtained from (10), (15) simply by assuming the existence on each link a second 1-cochain B and changing the first A by B:

$$S = k \sum_{M_3} B_\cup \delta A \tag{16}$$

Actually this action is very closely related to the general action (10). It is gauge invariant with respect to independent gauge transformations of B and A. Note also, that the formula (16) may evidently be extended to the case of a manifold of arbitrary dimension d provided B is considered a b-cochain and A - a (d-b-1)-cochain.

Let's consider now the lattice consisting of the cubic lattice and its dual one. Take the simplicial lattice obtained from it by connecting the vertexes of original lattice with its eight nearest vertexes of the dual one. The action for this lattice is given by (10), for one simplex - by (11), and the point is that the term in this action, containing the product of vertical and horizontal links (i.e. links of original cubic and its dual lattices) is gauge invariant by itself, since its variation consists of terms which can not emerge from other sources. One thus obtains the action

$$S = k \sum_{l-\text{links}} A(l)\delta A(l^{*}) \qquad (17)$$

defined for the lattice consisting of the cubic lattice and its dual one. The summation is over links of original lattice, and l^{*} is the plaquette dual to l. It is evident, that denoting the A on dual lattice by B and making some identification of links of dual lattice with the links of original one we re-obtain the action (16).

Action (17) seems to be the simplest one and is most useful for actual calculations. Note also that it may be written in the same form (17) but with original and dual lattices interchanged. In the continuum limit the actions (16) and (17) tend to the action

$$S \approx \int B_{\mu}\, \partial_{\nu} A_{\lambda}\, \varepsilon^{\mu\nu\lambda} \qquad (18)$$

which after changing the variables by B=Z+Y, A=Z-Y becomes the action for two Chern-Simons theories:

$$S \approx \int Z_\mu \, \partial_\nu Z_\lambda \, \varepsilon^{\mu\nu\lambda} - \int Y_\mu \, \partial_\nu Y_\lambda \, \varepsilon^{\mu\nu\lambda}$$

So, considering in the theories (16), (17), (18) the averages of quantities depending only on the sum A+B one obtains the answer for the Chern-Simons theory. Since cochains A and B in (17) lie on a different non-intersecting lattices this lattice representation of Chern-Simons theory introduces in a natural way a framing for Wilson loops. This point will be clarified in the next section.

Note also that the action similar to (17) may be obtained in the same way for all the lattices, consisting from an arbitrary lattice and its dual one.

The last point we wish to discuss here is the value of the coupling constant k in (10), (15)-(17). Denote by K a ring of coefficients in which the fields A(l) take values. K may be a ring of real numbers R, a ring of complex numbers C (in this case one has to modify (10) to obtain a real action), a ring of integers Z, a ring of integers modulo n – F_n, with n some positive integer, and so on. The gauge group of the theory is the same ring considered as an abelian group with respect to the summation operation. The quantity $\sum A \cup \delta A$ is an element of this ring. The action S is obtained from $\sum A \cup \delta A$ by using the ring homomorphism into the real numbers and multiplying by k. Thus, one has to choose the homomorphism. In the case K=R there is an evident choice, in the case $K=F_n$ the choice is also evident – the elements of F_n are represented by the integers 0,1,...,n-1. In this last case we have a possibility to choose k in the form $2\pi k/n$, with integer k and to consider A(l) as an element of Z (not F_n) since in the amplitude exp(iS) A(l) will be automatically taken modulo n. In the case K=Z also there exists an evident distinguished choice of homomorphism, the coefficient k in that case being an arbitrary real number which is essential only modulo 2π. Note also that the gauge group does not define the theory completely, since one can have a rings, which are isomorphic as abelian groups with respect to the sum operation, but differ in their multiplicative structure.

3. The observables and the skein relation.

In this section we shall calculate the averages of the gauge invariant observables of the theories - the Wilson loops. This will be done for both types of theories (BF and CS) and the calculation will clarify the lattice origin of the framing, which is important in continuous theory [5]. The main tool for the calculation the averages of Wilson loops is the skein relation [5] which connects the values of loops with overcrossing, undercrossing and reconnections. The skein relation is nothing but the loop equation for gauge theories (see [17]. In the abelian case this is shown below (this interpretation is correct also in continuous theory, as is evident from its derivation (within 1/k expansion) in [18]).

Consider case of the gauge group Z_n. The cochain A takes values in the ring (field for n simple) $F_n = \{0,1,\dots,n-1\}$, so $\exp((2\pi i/n)A)$ is an element of a group Z_n.
The Wilson loop is defined as

$$\Phi(C)=\exp((2\pi i/n)\sum_{l\in C} A(l)) \qquad (19)$$

Evidently, n-th power of $\Phi(C)$ is equal to 1. In principle one may consider the loops in representation with charge m, which means introducing the factor m in the exponent. In ›elian theory this is equivalent to considering m loops with unit charges, so we shall always take $m=1$.

The average $\langle \Phi(C_1)\ldots\Phi(C_r)\rangle$ is given by

$$(1/Z) \int \prod_{l} dA(l) e^{iS(A)} \prod_{i} \Phi(C_i) \qquad (20)$$

where l runs over all the links of the lattice, and Z is the partition function.

For the theories (10) on general irregular lattice the averages (20) are not smooth in a sense that their values may change strongly for a small variations of loops, even when this variation doesn't change the topology of loops. To see that let's consider the integration in (20) over A(l) for some link l (for simplicity take coefficient $k=1$ in the action). Each tetrahedron containing the link l will contribute to the coefficient in front of A(l) in the action (10) a value A(l'), where l' is its only link having no common points with l (remember (11). All such links l' form a closed contour which we denote l^*. In Fig.1 l' are shown in the case when there are only three tetrahedrons containing link l. The integration over A(l) gives $\delta(F(l^*))$ in the case when there are no Wilson lines through l and, for example, $\delta(F(l^*)+1)$ when there is one Wilson loop of minimal charge, passing through l. Imagine now that we have only one Wilson loop C which may be represented as a sum of loops $(l_i)^*$ for the links l_i, $i=1,2,\ldots$ (we assume also that set of links $\{l_i\}$ has no common elements with links in the set $\{(l_i)^*\}$). Then the average (20) for such loop is 1, since that loop disappears from the integrand after integration over all l_i. On the other hand, assume that there exists some other link

l such that $(l)^* = l_i^*$ for some i, say $i=1$. Then if we change slightly our loop in such a way that it still doesn't contain l_1, but pass through l, then integrations over l_1 and l give contradictory delta-functions, and the answer for (20) is zero. An example for such a situation with $C = l^* = (l_1)^*$ may be found from Fig.1 if we imagine that the tetrahedron is glued over the face l^* with some other similar tetrahedron, with link l_1 being symmetrical to l. In the same way the example may be constructed for another contours C. So, in the continuum limit the value of (20) may change strongly under a small variations of loops which means that (20) has no smooth continuum limit. We shall see below that if we consider the representation of Chern-Simons theory through the BF-system mentioned at the end of last paragraph, a smooth limit exists. In other words, only the framed loops on the lattice have a continuous limit.

Consider now the theory with action (17) (with coefficient $k \to 2\pi k/n$, with integer k) on a lattice, consisting from a cubic lattice and its dual one (nothing changes for the case of lattice, consisting from some arbitrary one and its dual). Consider first the case, when all the loops lie on a one lattice, not on the both mutually dual ones. In this case we integrate over the links of dual lattice and obtain the multiply of delta-functions of the field strengths on the plaquettes of original lattice:

$$(1/Z) \int \prod_l dA(l) \prod_p \delta(kF(p)) \prod \Phi(C_i) \qquad (21)$$

where l and p run over all the links and plaquettes, respectively, of the original lattice, and $F(p) = \delta A(p)$. Symbol $\delta(kF(p))$ is actually Kroneker delta which requires $kF(p)=0$ (mod n).

If we assume that k and n have no common integer divisors, then $\delta(kF(p))$ means that $F(p)=0$. After that it is evident that all the loops may be continuously deformed without changing the value of the integral. So, the contractable loops may be shrunk to a point and hence trivially disappear from the integrand. For non-contractible loops we use the lattice version of Stokes theorem

$$\sum_{\partial S} A = \sum_{S} \delta A = \sum_{S} F(p) \tag{22}$$

to deduce the fact that actually only the homological class of the given collection of loops is essential. Remembering that n-th power of loop is equal to 1, we see that the homological classes in question are with coefficients in F_n, i.e. $H_1(M,F_n)$. If the sum of the elements of these classes, defined by each loop, is zero, i.e. the integrand in (21) doesn't depend on A, then the answer for (21) is 1, otherwise the integration in (21) gives zero answer.

The general answer for a collection of loops on both lattices may be found in the same way by direct calculation. It is however much more transparent to derive a skein relation for this theory which permits one to disentangle the loops and hence express the answer through the answer for a collection of unknotted, unlinked loops.

We turn now to the derivation of loop equation for this theory. Consider the initial integral (20) and take a link p on one of the lattices. Due to relation

$$\int \prod_{l} dA(l)\ \{f(A(p)+a)-f(A(p))\} = 0$$

where a is an arbitrary element of ring, we obtain for f(A) :

$$e^{iS(A)}\prod_i \Phi(C_i) :$$

$$\langle\Phi(C_1)\ldots\Phi(C_r)\rangle =$$
$$\langle\Phi(C_1)\ldots\Phi(C_r)e^{ak(2\pi i/n)\delta A(p^*)}\rangle \; e^{(2\pi i n(p)a/n)} \qquad (23)$$

where p^* is the plaquette dual to link p, and n(p) is the algebraic number of Wilson lines passing through p. The quantity $e^{ak(2\pi i/n)\delta A(p^*)}$ is equal to $(\Phi(p^*))^{ak}$ i.e. the Wilson line with charge ak (mod n) appears. In the case, when k and n have no common integer divisor (this is always the case if n is the prime number) we can always find a such that ak=1 (mod n). In this case relation (23) applied to different links p may be used to change the configuration of loops in an arbitrary way. In particular, it gives the relations

$$L_+=L_-\exp\{(2\pi i a(k)/n)\} \qquad (24)$$

where $L_+(L_-)$ denotes overcrossing (undercrossing) of loops, one of which is on the original lattice and second on a dual one. Together with the property that the loops on one lattice don't feel each other (this also follows from (23) at appropriate choice of link p) , the skein relation (24) permits one to reduce the calculation to that for the collection of unknotted, unlinked loops. The answer for this last configuration may be found directly from definition (20). This answer is given above for the case, when all the loops are on one of the lattices. In the case of loops on both lattices the answer is zero in all the cases except the one when sum of elements of $H_1(M,F_n)$, defined by each loop, is zero separately for each lattice. In this last case the answer for (20) is given by the product of factors in the r.h.s. of (24), which arise every time when (24) is used to disentangle the loops. As a simple example consider two homotopically trivial loops C_1, C_2 (on the original and dual lattices, respectively). The average (18) for this configuration is equal to $\exp(2\pi i N(C_1,C_2)a(k)/n)$, where $N(C_1,C_2)$ is the linking number, taken modulo n, of C_1 and C_2. This is in correspondence with the fact that Z_n is "smaller" that $U(1)$, and corresponding knot invariants are weaker.

According to the end of Sect.2, one has to represent the Wilson loops of continuous theory by the pair of loops – A-loops and B-loops – in the lattice theory (17). Since in the continuous limit these loops have to tend to each other to obtain the sum A+B in one point, it means that the distance between A- and B-loops, representing one continuous loop, has to be much smaller than the distance between different continuous loops. This means that the pair of A- and B-loops on the lattice is the lattice counterpart of the framed continuous loop.

Above we constantly consider the case of mutually simple k and n. In the case when there exist the non-trivial common integer divisor q of k and n the answer for (20) is the following: if in (20) enter only the loops with the charges $q_i=c_iq$, (i.e. proportional to q), then the answer evidently reduces to that for the theory with $n\to(n/q)$, $k\to(k/q)$ and loops with charges c_i. If not all the loops have charges proportional to q, then answer is zero. This may be proved easily: consider (21) and take the link l with Wilson loop with charge, which is not proportional to q. Then we parameterize $A(l)$ as $A(l)=r+s(n/q)$ ($r=0,\dots,(n/q)-1$, $s=0,\dots,q-1$) and the sum over $A(l)$ may be represented as the sum over r and s. It is not difficult to understand, that the sum over s, at fixed r, is zero due to the relation $\sum_s \exp(2\pi is/q)=0$.

A few words concerning the derivation of complete skein relation for the ring K=R. The quantity a in this case is an arbitrary real number and there are no restrictions on derivability of skein relations.

4. The manifolds with boundaries. Connection with two-dimensional statistical systems.

In this section we shall consider the theory (17) on the three-dimensional manifold $M^3=D^2xR^1$ with D^2 being a two-dimensional disk and establish its connection with two-dimensional statistical systems in a way similar to that of Refs. [5,19]. Again the convenient lattice is the lattice, consisting from two mutually dual regular cubic lattices and it will be useful (and certainly harmless) to denote the fields on both the original and dual lattices by the same letter A. The boundary of the manifold $M^3=D^2xR^1$ is the cylinder $\partial M^3=S^1xR^1$. M^3 is composed of cubes of original lattice and we take the dual lattice such that its boundary lies inside the original lattice. We also subject the field A to the boundary condition saying that its value on the vertical (i.e. along R^1) boundary links of the original lattice is zero. The formulae below are written down for the case $K=F_n$, although they are valid also in the case K=R. Let's integrate now over all the vertical links of both lattices. As a result, we obtain a collection of delta-functions $\delta(kF(p)$, as in (21), except that now plaquettes p are all the horizontal ones of both lattices. As a solution of all these delta-functions constraints we find, in the case of mutually simple k and n, that at given "time" (i.e. R^1) slice cochain A is exact:

$$A = \delta^{(2)}\phi \qquad (25)$$

Now we have to substitute this back into the action. The remaining part of the action i.e. the part not containing vertical links is:

$$S' = (2\pi k/n) \sum_{l} A(l)(A(l^+) - A(l^-)) \qquad (26)$$

where the sum is over all the horizontal links l of dual lattice, and the locations and the orientations of link l and links l^+, l^- of the original lattice is drawn on Fig.2. Substituting (25) into (26) we see, that the action depends only on the fields on the boundary of M_3 :

$$S = (2\pi k/n) \sum \pm \phi_i \phi_j \qquad (27)$$

where the sum is over nearest neighbours of the two-dimensional lattice, obtained by connecting the nearest vertexes of the boundaries of original and dual lattices (Fig.3). In another words, the new lattice is obtained by: (a) taking boundary of the original lattice (forgetting the dual one) (b) connecting the center of each plaquette with its vertexes, (c) removing the links of original lattice.

The sign in (27) is + for interaction in x+y direction and - for the interaction in the x-y direction (Fig.3). Denoting the coordinates on the rotated by $\pi/4$ coordinate system by (x^+,x^-) (Fig.3) one may rewrite the action (27) in the form

$$S = (2\pi k/n) \sum \phi(x^+,x^-)\, \phi(x^++1,x^-) - \phi(x^+,x^-)\, \phi(x^+,x^-+1) \qquad (28)$$

The pathology with signs may be cured by reversing the sign of $\phi(x^{+},x^{-})$ for even x^{-}. (28) may be considered as the action of two-dimensional statistical system.

For the case of gauge group Z_2 one can go further and make the following field redefinition: instead of variables ϕ_i, taking values 0,1, we introduce the variables $\sigma = 1-2\phi$, taking values ±1. Substituting into the (27) and making the necessary sign adjustment one obtains the action for the two-dimensional Ising model at a given point.

The authors are grateful to An.R.Kavalov for discussions. Al.K. would like to thank A.Sedrakyan for useful comments.

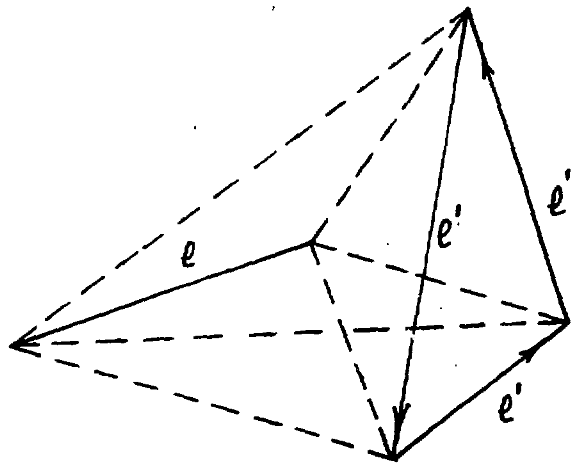

Fig.1

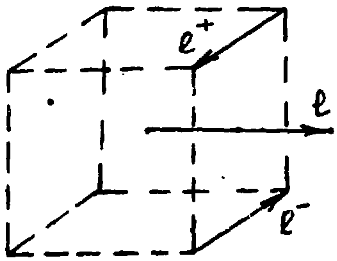

Fig.2)

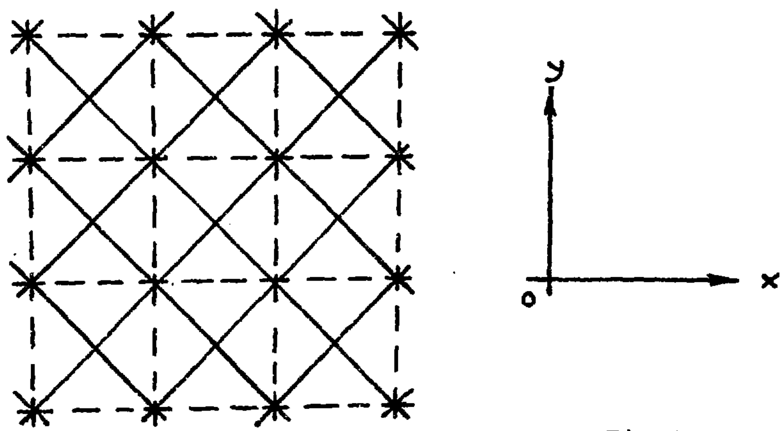

Fig.3

References

1. Al.R.Kavalov and R.L.Mkrtchyan. The lattice construction for abelian lattice gauge theory, preprint YERPHY-1219(5)-89, Yerevan 1989; Phys.Lett.B (in press)
2. A.S.Schwarz. The partition function of degenerate quadratic functional and Ray-Singer invariants. – Lett.Math.Phys., 1978, vol.2, p.247.
3. A.S.Schwarz. The partition function of degenerate functional. Comm.Math.Phys., 1979, vol.67, p.1.
4. A.M.Polyakov. Fermi-Bose t~ ~smutations induced by gauge fields. – Mod.Phys.Lett 1~ 8, vol.A3, p.325.
5. E.Witten. Quantum field theory and the Jones polinomials. Comm.Math.Phys., 1989, vol.‥21, ‚ 351.
6. E.Witten. 2+1 dimensional gravity .s an exactly soluble system. Nucl.Phys., 1988/89, vol.B311, p.46-78.
7. G.T.Horowitz. Exactly soluble diffeo~orphysm invariant theories Santa-Barbara preprint ITP-NSF-‚ -178.
8. M.Blau and G.Thompson. Topological gauge theories of antisymmetric tensor fields. – PAR LPTHE 89-18, S.I.S.S.A. 44/89/FM, Paris.
9. M.Blau and G.Thompson. A new class of topological field theories and the Ray-Singer torsion. – Phys.Lett., 1989, vol.B228, p.64.
10. J.F.Wheater. A topological lattice model. – University of Oxford preprint OUTP-89-08P, February 1989
11. V.A.Kazakov, I.K.Kostov and A.A.Migdal. Phys.Lett., 1985, vol.157B, p.295.
12. V.A.Kazakov and A.A.Migdal. Nucl.Phys., 1988/89, vol.B311, p.171.

13. F.David. Nucl.Phys., 1985, vol.257, p.45.

14. J.Ambjorn, B.Durhus and J.Frohlich. Nucl.Phys., 1985, vol.B257, p.433.

15. V.G.Knizhnik, A.M.Polyakov and A.B.Zamolodchikov. Mod.Phys.Lett., 1989, vol.A3, p.819.

16. R.Dijkgraaf and E.Witten. Topological gauge theories and group cohomology, preprint IASSNS-HEP-89/33, THU-89/9 (1989)

17. A.A.Migdal. Loop equations and 1/N expansion Phys.Rep., 1983, vol.102, p.199.

18. P.Gotta-Ramusino, E.Guadagnini, M.Martellini and M.Mintchev CERN-TH-5277/89, IFUP-TH 1/89 (1989).

19. G.Moore and N.Seiberg. Taming the conformal zoo, Phys.Lett. 1989, vol.B220, p.422.

20. A.Dold. Lectures on algebraic geometry. - Springer-Verlag 1972.

The manuscript was received May 22, 1990

АЛ. Р. КАВАЛОВ, Р. Л. МКРТЧЯН

РЕШЕТОЧНАЯ АБЕЛЕВА КАЛИБРОВОЧНАЯ ТЕОРИЯ ЧЕРНА-САЙМОНА

(на английском языке, перевод авторов)

Редактор Л. П. Мукаян

Технический редактор А. С. Абрамян

Подписано в печать 2/VII-90 ВФ-03615 Формат 60x84x16
Офсетная печать. Уч. изд. л. 1.0 Тираж 299 экз. Ц. 15 к.
Зак. тип. 188 Индекс 3649

Отпечатано в Ереванском физическом институте
Ереван-36, ул. Братьев Алиханян 2.

The address for requests:
Information Department
Yerevan Physics Institute
Alikhanian Brothers 2,
Yrevan, 375036
Armenia, USSR

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ