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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
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# GEOMETRY OF MATHEMATICAL FRACTALS



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Ի.Ա.ԲԱՂԴԱՍԱՐՅԱՆ

# ՄԱԹԵՄԱՏԻԿԱԿԱՆ ՖՐԱԿՏԱԼՆԵՐԻ ԵՐԿՐԱՉԱՓՈՒԹՅՈՒՆ

Կառուցված է նոր երկրաչափություն՝ կապված ֆրակտալների հետ, որի ժամանակ այդպիսի երկրաչափական հասկացողություններն, ինչպիսին են մետրիկան, ծավալի չափը, ձևափոխությունների խումբը և այլն, անընդհատորեն կախված են, այսպես կոչված, ֆրակտալի կառուցման պարամետրից։ Մի քանի ֆրակտալների հիման վրա ցույց է տրված, թե ինչպես այդ երկրաչափությունը կարող է վերջավոր դարձնել այն երկրաչափական չափերը, որոնք առաջվա երկրաչափությունում հավասար էին զրոյի կամ էլ անվերջության, ընդ որում ֆրակտալի և սկզբնական բազմության, որի հիման վրա էլ կառուցվում է տվյալ ֆրակտալը, չափողականությունների տարբերության զրոյական ձգտման դեպքում (բնական, եթե հնարավոր է այդ սահմանային անցումը) նրանք ձգտում են սկզբնական բազմություններին համապատասխանող երկրաչափական չափերին

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# ГЕОМЕТРИЯ МАТЕМАТИЧЕСКИХ ФРАКТАЛОВ

Применительно к фракталам построена новая геометрия, в которой такие геометрические понятия, как метрика, мера объема, группа преобразований и так далее непрерывны относительно так называемого параметра построения фрактала. На примере нескольких фракталов показано, каким образом эта геометрия может приводить к конечным значениям геометрических мер фракталов, которые в прежней геометрии оказывались равными нулю или бесконечности, причем при стремлении к нулю разности между размерностью фрактала и размерностью начального множества, на основании которого строится фрактал (разумеется, при наличии возможности такого предельного перехода), они стремятся к геометрическим мерам соответствующих начальных множеств.

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# GEOMETRY OF MATHEMATICAL FRACTALS

In conformity with fractals, a new geometry is constructed, in which such geometrical concepts as metrics, volume's measure, transformation group, etc., are continuous relative to the so-called fractal building parameter. It is shown on the example of some fractals how this geometry can result in finite values of fractal's geometrical measures that were proved to be zero or infinite in the former geometry and besides, when the difference between the dimension of the fractal and that of the initial set on which this fractal is built up approaches zero (of course, if there is a possibility of such limit transition), they aim at geometrical measures of corresponding primary sets.

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## 1. INTRODUCTION

The so-called mathematical fractals [1,2] are the sets generated as a result of an infinite interpolation with dimensions that in general differ from those of initial sets on which they are built up. Application of usual Euclidean geometry to them leads in most cases to zero or infinite values of geometrical measures of these sets, except for the so-called "thick fractals" [1,2].

As an example, let us consider the Cantor set obtained from the segment [0,1] by throwing out the q-th part of every segment got on the previous step (Fig.1).

The Hausdorff's dimension [1,2] of this set is

$$d = \ln 2/[\ln 2 - \ln(1-q)] , \qquad (1.1)$$

whereas the length L(q) that represents the sum of lengths of segments remaining after infinite discarding, is

$$L(q) = 1 - q - (q/2)(1 - q)2 - \ldots = 1 - q(1/q) = 0 \qquad (1.2)$$

It is noteworthy that the length L(q) is equal to zero identically for any $q>0$, suffering rupture in the point $q=0$, because $L(0)=1$ and does not actually distinguish between different q.

Let us now turn to the fractal curve (we call it "continuous Cantor curve", since it is built up on the basis of the Cantor set (Fig.1)) derived again from the segment [0,1] according to Fig.2, all the segments of the broken line being taken equal amongst themselves. The Hausdorff's dimension of the curve is

$$d = 2\ln 2/[\ln 2 - \ln(1 - q)] \qquad (1.3)$$

And the length L(q) turns out to be infinite

$$L(q) = \lim_{k\to\infty} [2(1 - q)]^k = \infty , \qquad (1.4)$$

since in accordance with the construction $0<q<1/2$. Thus, in the case of the continuous Cantor curve too, the length L(q) being determined in the usual way, does not distinguish between fractals that correspond to different values of q and besides, suffers a rupture at $q=1/2$ as $L(1/2)=1$.

The self-similar fractal shown in Fig.3 should also be mentioned (we call it "ruptured Cantor curve"), the dimension of which for the same value of $q>0$ coincides with the dimension of the continuous Cantor curve (1.3), but at the same time, it has an absolutely different topological structure. As it is seen, in this case even the usual Hausdorff's dimension does not distinguish between these two quite different self-similar fractals. As to the length determined in the usual way, it is infinite for both of the fractals (at $0<q<1/2$).

But if for a stronger geometrical distinguishing we want to extend the measures of the corresponding prefractal partly-smooth geometrical objects to fractals themselves (e.g., in case of the Cantor set shown in Fig.1 this measure is equal to the length), which are continuous relative to the building parameter (what is q for fractals plotted in Figs. 1-3), then we ought to change the geometry of the space around the fractal in such a way, that the geometrical measure of the corresponding prefractal objects in the limit of the infinite interpolation should tend to a finite value not equal in



general to zero (even in case of Cantor sets with $0<d<1$ [1], where d is the fractal dimension), since only in this case there is a possibility for the limit transition $q\to q_0$ ($q_0$ corresponds to the initial set) after infinite interpolation. Now we will construct the eigengeometry of just this fractal.

## 2. EIGENGEOMETRY OF FRACTAL

Before constructing the geometry of a mathematical fractal, note that no finite-dimensional space will do for this, if required that the volume measure of the neighbourhood of an arbitrary point of a fractal that is in a power dependence with the diameter of the neighbourhood with index d defined as the fractal's dimension, should be represented as a product of increments of coordinates at a given point of this space (since it is known that the volume of a range in the n-dimensional space given in this way is proportional to the n-th degree of the diameter of this range) and beside this, at the limit $d\to d_0$, where $d_0$ is the dimension of the initial set, without additional limitations on the group of admissible transformations (see below), and correspondingly on the space on which it acts, all geometrical values should tend continuously to their $d_0$-dimensional analogs.

Here Refs.[3,4] should be also mentioned, where the infinite dimensionality of the space put in correspondence with the dimension d, but in some other aspect, necessarily follows from the postulated relations laid on the metric tensor.

Proceeding from the aforesaid, let us define the infinite space with basis vectors $\{\vec{e}_i(u)\}_{i=1}^{\infty}$, where $u=(u^1,u^2,\dots)$ are coordinates of a point in this space. Correspondingly, let us define the differential of the vector $d\vec{u}$ and the metrics $ds^2$ as

$$d\vec{u} = du^i\,\vec{e}_i(u) \qquad (2.1)$$

$$ds^2 = G_{ij}(u)\,du^i du^j \qquad (2.2)$$

where $G_{ij}(u)$ is the metric tensor. Repeating indices are meant to be summed from 1 to $\infty$. Both, the basis vectors and the metric tensor are assumed to be dependent of d and $d-d_0$. Since

$$G_{ij}(u) = \vec{e}_i(u)\vec{e}_j(u) \ , \qquad (2.3)$$

it follows from invariance of $ds^2$ under transformations

$$\vec{e}'_j(u') = \frac{\partial u^i}{\partial u'^j}\,\vec{e}_i(u) \ , \qquad (2.4)$$

that

$$G'_{ij}(u') = \frac{\partial u^k}{\partial u'^i}\,\frac{\partial u^l}{\partial u'^j}\,G_{kl}(u) \ . \qquad (2.5)$$

We define the volume measure invariant under (2.4) in the given space as:

$$dV_u = \lim_{N\to\infty} \prod_{m=1}^{N} \frac{du^m}{\Lambda_m^{\rho_m}} \sqrt{G(u)} \qquad (2.6)$$

where G(u) is the determinant of the infinite-dimensional matrix $G_{ij}(u)$; $\Lambda_m>0, \rho_m>0$ depend on d and $d-d_0$, by the way, $\Lambda_m$ have the same dimensions as $u^m$; $\rho_m$ are dimensionless and, as by definition the dimension of the volume in (2.6) is equal to d, then

$$\lim_{N\to\infty} (N - \sum_{m=1}^{N} \rho_m) = d \ . \qquad (2.7)$$

Let us demand that at the limit $d\to d_0$ (implying a continuous dependence of the dimension d on the fractal's building parameter q) all of expressions (2.1)-(2.6) should turn continuously into their $d_0$-dimensional analogs. For this purpose let us choose the metric tensor $G_{ij}(u)$ in the form

$$G_{ij}(u) = \bar{G}_{ij}(u)\cdot\omega_i(u)\cdot\omega_j(u) \ , \quad \text{(n.s.)} \qquad (2.8)$$

where (n.s.) means "no summing", $\bar{G}_{ij}(u)$ is a sufficiently arbitrary matrix turning at the limit $d\to d_0$ into $\delta_{ij}$, and $\omega_m(u)$, $\Lambda_m(d,\Delta)$, $\rho_m(d,\Delta)$ functions, where $\Delta=|d-d_0|$, have the following properties at $d\to d_0$:

$$\omega_m(u) \longrightarrow \begin{cases} 1, & m\le d_0 \\ 0, & m>d_0 \end{cases} \qquad (2.9)$$

$$\omega_m(u)/(\Lambda_m^{\rho_m}) \longrightarrow \begin{cases} 1, & m\le d_0 \\ \delta(u^m), & m>d_0 \end{cases} \qquad (2.10)$$

where $\delta(u^m)$ is the $\delta$-function that depends on the coordinate $u^m$. Furthermore, the form of the metric tensor (2.8) remains the same after the following general coordinate transformations

$$u'^i = F^i(\phi_1(u^1), \ldots, \phi_5(u^5), \ldots) \ , \qquad (2.11)$$

where

$$\phi_s(x) = \int_0^x \omega_s(t)dt, \qquad s\ge 1 \qquad (2.12)$$

and $F^i(x^1,\ldots)=0$ at $d=d_0$ for $i>d$. Thus, at the limit $d\to d_0$ the continuous transition of expressions (2.1)-(2.5) into their corresponding $d_0$-dimensional analogs is secured. The measure (2.6) consists of the product of $\delta$-functions depending on the coordinates $u^m$ for $m>d$, and after integrating on a range that includes points with $u^m=0$ for $m>d_0$, turns into the measure of this $d_0$-dimensional set.

In this geometry, within the framework of group-forming transformations (2.11) any tensor $T_{i_1 i_2 i_3 \dots i_p}(u)$ given on the fractal has the form:

$$T_{i_1 i_2 i_3 \dots i_p}(u) = \bar{T}_{i_1 i_2 i_3 \dots i_p}(u)\, \omega_{i_1}(u)\omega_{i_2}(u)\dots\omega_{i_p}(u) \qquad (2.13)$$

where $\bar{T}_{i_1 i_2 i_3 \dots i_p}(u)$ are sufficiently arbitrary functions. Since for any 2-valent tensor $T_{ij}(u)$ that continuously depend on d according to (2.13) we have

$$\det(T_{ij}(u)) = \det(\bar{T}_{ij}(u))\cdot\Big(\prod_{m=1}^{\infty} \omega_m(u)\Big)^2 , \qquad (2.14)$$

then requiring, that in correspondence with continuity of the matrix $T_{ij}(u)$

$$\lim_{\substack{d \longrightarrow d_0 \\ 1\le i,j\le\infty}} (T_{ij}(u)) = \lim_{\substack{d \longrightarrow d_0 \\ 1\le\alpha,\beta\le d_0}} (\bar{T}_{\alpha\beta}(u)) \qquad (2.15)$$

there should be continuity relative to d also for $\det(T_{ij}(u))$, we derive the relation

$$\lim_{d\to d_0} \det(\bar{T}_{ij}(u))\cdot \lim_{d\to d_0} \Big(\prod_{m=1}^{\infty}\omega_m(u)\Big)^2 = \lim_{d\to d_0} \det(\bar{T}_{\alpha\beta}(u)) \qquad (2.16)$$

from which the necessity of the infinite dimensionality of the space put in correspondence with the neighbourhood of the fractal is evident, because in the finite-dimensional case the equality analogous to (2.16) is in general not true because of the property (2.9) of functions $\omega_m(u)$ and arbitrariness of functions $\bar{T}_{ij}(u)$. Then it follows from (2.16) (also through the arbitrariness of functions $\bar{T}_{ij}(u)$) that



$$\lim_{d\to d_0} \Big(\prod_{m=1}^{\infty} \omega_m(u)\Big) \ne 0,\infty . \qquad (2.17)$$

An example of $\omega_m(u)$, $\Lambda_m(d,\Delta)$ and $\rho_m(d,\Delta)$ functions that satisfy the relations (2.7), (2.9), (2.10) and (2.17) is given in the Appendix.

By integrating (2.6) over $-\varepsilon^m/2\le u^m\le\varepsilon^m/2$, $m\ge1$ segments, where $\varepsilon^m$ meet the condition

$$\lim_{m\to\infty} |\varepsilon^m|/\Lambda_m = 1, \qquad d\ne d_0 , \qquad (2.18)$$

ensuring (along with the requirements $\lambda_m^2\to0$, $\lambda_m^2\cdot\tau_m^2\to\infty$ at $m\to\infty$, see Appendix) the convergence of the volume measure (2.6), and passing to values $\varepsilon=\max|\varepsilon^j|$, $1\le j\le\infty$ and $\nu^m=\varepsilon^m/\varepsilon$, at $\varepsilon\sim0$ we obtain

$$\Delta V \sim \varepsilon^d F_d(\varepsilon) , \qquad (2.19)$$

where $F_d(\varepsilon)$ is a finite function at $\varepsilon\to0$. Then let us cover the fractal with a minimal number of ranges of the volume given by the expression (2.19) which are necessary for its enclosure in the infinite-dimensional Riemannian space. Let us demand that the sum

$$\Delta V = \sum_{a=1}^{N} \Delta V_{\varepsilon_a} \qquad (2.20)$$

should tend to a finite value at $\max \varepsilon_a\to0$, $1\le a\le N$, where $\varepsilon_a = \max|\varepsilon_a^m|$, $1\le m\le\infty$, $\varepsilon_a^m$ are local increments of coordinates in the infinite-dimensional space, N is the total number of ranges. Then it is easy to show that

$$\lim_{\substack{\max \varepsilon_a\to0 \\ 1\le a\le N}} [\ln N - \ln K + \overline{\ln(\varepsilon^d)}] = 0 , \qquad (2.21)$$

where K tends to a constant not equal to 0 or $\infty$, $\overline{\varepsilon^d}$ is the d-th power of values averaged over all ranges.

Since in the infinite interpolation limit the neighbourhood of the fractal is put in correspondence with the infinite-dimensional Riemannian space, the coordinates $x_k^{\alpha}$ ($1 \le \alpha \le n$) of points of the prefractal partly-smooth geometrical object at $k \ll k_0$, where k is current number of interpolation, and $k_0$ is a critical number at which the geometry of the space near the fractal starts changing, in the primordial Euclidean space with dimension n (unambiguously bound up with $d_0$) and the coordinates $u_k^m$ of these points in the intermediate Riemannian space are coupled as follows:

$$u_k^m = g_k^m(x_k^1, \ldots, x_k^n); \qquad m \ge 1 \tag{2.22}$$

by the way, at $k \ll k_0$

$$u_k^m \simeq \begin{cases} x_k^m, & 1 \le m \le n \\ 0, & m > n \end{cases}, \tag{2.23}$$

and also

$$\lim_{q \to q_0} u_k^m = \begin{cases} x_k^m, & 1 \le m \le d_0 \\ 0, & m > d_0 \end{cases}, \tag{2.24}$$

$$\lim_{k \to \infty} u_k^m = u^m, \quad m \ge 1, \tag{2.25}$$

where the value of the parameter $q=q_0$ corresponds to the dimension $d=d_0$. As to the connection between the metric tensor of the primary n-dimensional Euclidean space $\delta_{\alpha\beta}$ and that of the limit infinite-dimensional Riemannian space $G_{ij}(u)$, it is realised by the intermediate metric tensor $G_{ij}^k(u)$ which at $k \ll k_0$



must be

$$G_{ij}^k(u) \simeq \begin{cases} \delta_{ij}; & 1 \le i,j \le n \\ 0; & i,j > n \end{cases}, \tag{2.26}$$

and furthermore,

$$\lim_{q \to q_0} G_{ij}^k(u) = \begin{cases} \delta_{ij}; & 1 \le i,j \le d_0 \\ 0; & i,j > d_0 \end{cases}, \tag{2.27}$$

$$\lim_{k \to \infty} G_{ij}^k(u) = G_{ij}(u). \tag{2.28}$$

Since starting with some critical interpolation number k the geometry is changing according to (2.23), (2.25), (2.26) and (2.28), for self-consistency it is necessary to change the interpolation process itself. For example, if in case of the Cantor set (Fig.1) at $k \ll k_0$ the middle parts of segments coinciding with geodesics in the primordial Euclidean space ($n=1$) are discarded, at $k \ge k_0$ the "middle" parts of geodesics no longer coinciding in general with segments of the initial set [0,1], but being in one-one correspondence with them according to (2.22), are discarded.

Assuming a sufficiently partly-smooth behaviour of $g_k^m(x_k^1, \ldots, x_k^n)$, we have

$$\overline{\varepsilon^d} = \Gamma \cdot \overline{\delta^d}, \tag{2.29}$$

where $\Phi$ is a finite value not equal to 0 at $\max \varepsilon_a \to 0$, $1 \le a \le N$, $\overline{\delta^d}$ is the average d-th power of $\delta_a$ that represent the lengths of ribs of n-cubes covering the prefractal partly-smooth geometrical object in the initial Euclidean n-dimensional space.

With account of (2.29), from (2.21) we derive the following expression for the fractals corresponding to self-similar prefractal objects in the initial space

$$d = \lim_{\delta \to 0} \ln N(\delta)/\ln(\delta^{-1}) \qquad (2.30)$$

that coincides with the definition of the Hausdorff's dimension.

It should be added in conclusion, that the infinite-dimensionality of the space put in correspondence with the neighbourhood of the fractal, is conditioned also by the fact that fractals with dimensions of initial Euclidean spaces tending directly to $\infty$ where prefractal objects are disposed, can correspond to the same value of d, e.g., the same Cantor curve (Fig.2), but every next interpolation step of which is carried out in a new dimension. Thus, the most general space corresponding to $d \neq d_0$ is infinite-dimensional.

### 3. CALCULATION OF GEOMETRICAL MEASURES OF FRACTALS

So, the requirement of fractal measure finiteness (the thing is about the prefractal partly-smooth object's measure, such as length, area, etc., referring to the fractal itself in the limit of the infinite interpolation) provides the fractal's neighbourhood with properties of a Riemannian space which turns out by necessity infinite-dimensional. Furthermore, the partly-smooth behaviour of $g_k^m(x_k^1, \ldots, x_k^n)$ functions in (2.22) for any $k \geq 1$ keeps characteristic features of the construction for the fractal's projection onto the initial Euclidean space (i. e., e.g., there is a self-similarity for the prefractal object, it will be kept for the projection of the fractal in the limit of infinite interpolation). Therefore, it is possible to calculate independently the dimension d of the fractal and its geometrical measure. As to elements of the length, area, etc., then when calculating the corresponding measure, one is



to use their Riemannian generalizations. On the example of some fractals we show how the new geometry allows to obtain finite values for geometrical measures of fractals.

Let us consider at first the fractals, the measures of which are lengths. In correspondence with the aforesaid, the shortest lines between the points formed on the k-th step of the interpolation, where $k \geq k_0$ and $k_0$ is a critical number of interpolation in the initial Euclidean space with n dimension, are not the straight lines that connect these points, but the geodesics, since the space around them is no longer Euclidean. According to (2.2), taking into account (2.22)-(2.28), the fractal curve length between $P_1$ and $P_2$ points is defined as:

$$L_d = \min \lim_{M_k \to \infty} \sum_{a=1}^{M_k} \int \sqrt{G_{ij}^k(u_k^m)\, du_k^i\, du_k^j}\ , \qquad (3.1)$$

where $M_k$ is the number of links $E_a^k$ of the partly linear prefractal curve between $P_1$ and $P_2$ points formed on the k-th step of interpolation, and the minimum is taken because of the fact that the sections of straight lines in the initial Euclidean space are put in correspondence with the geodesics in the infinite-dimensional Riemannian space around the fractal.

In particular, from (3.1) for the Cantor set (Fig.1) we obtain

$$L_d^{cs} = \min \lim_{M_k \to \infty} \exp(-\varkappa \ln M_k) \cdot \sum_{a=1}^{M_k} F_a(M_k, \varkappa)\ , \qquad (3.2)$$

where

$$M_k = 2^k\ ; \quad F_a(M_k, \varkappa) = \sqrt{G_{ij}^k(u_k^m(x_{ka}^1))\, v_{ka}^i\, v_{ka}^j}\ ,$$

$$v_{ka}^i = \partial u_k^i(x_{ka}^1)/\partial x_{ka}^1\ ; \quad \varkappa = 1/d = 1 - \ln(1-q)/\ln 2\ . \qquad (3.3)$$

It is easy to show that

$$(\ln N)^l = \sum_{a=1}^{N} a^{-\alpha_l(N)} - C_{lN} \qquad (3.4)$$

for any real number $l \geq 1$ and sufficiently large $N$, where

$$\lim_{N\to\infty} C_{lN} = C$$

$$0 < \alpha_l(N) \leq 1 \quad (= 1 \ \text{at} \ l=1) \qquad (3.5)$$

$$\alpha_l(N) \sim 1 - 1/(\ln N)^l \ ; \quad N \gg 1, \quad l > 1 ,$$

and $C=0.5772...$ is the so-called Euler's constant. The relation (3.4) generalizes the equality

$$\lim_{N\to\infty} \left(\sum_{a=1}^{N} 1/a - \ln N\right) = C$$

which is well-known from the mathematical analysis. Demanding finiteness of $L_d^{cs}$ in (3.2) and using (3.4), we obtain a general expression for $F_a(M_k, \varkappa)$

$$F_a(M_k,\varkappa) = [H_a(M_k,\varkappa) - D_a(M_k,\varkappa)]\, \Psi(M_k,\varkappa) \ , \qquad (3.6)$$

where

$$H_a(M_k,\varkappa) = \sum_{l=1}^{L_{M_k}} (\varkappa^l/l!)\cdot a^{-\alpha_l(M_k,\varkappa)} \qquad (3.7)$$

$$D_a(M_k,\varkappa) = C_a(M_k,\varkappa) - E_a(M_k,\varkappa) - 1/M_k \qquad (3.8)$$

$$C_a(M_k,\varkappa) = \sum_{l=1}^{L_{M_k}} (\varkappa^l/l!)\cdot C_{al}(M_k)$$



$$C_{lM_k} = \sum_{a=1}^{M_k} C_{al}(M_k)$$

$$\lim_{M_k\to\infty} \exp(-\varkappa \ln M_k)\cdot \sum_{a=1}^{M_k} E_a(M_k, \varkappa) = 0$$

$$L_d^{cs} = \min \lim_{M_k\to\infty} \Psi(M_k, \varkappa) \qquad (3.9)$$

$L_N$ is the maximal power of the logarithm for given $N$ at which the relation (3.4) is true, and for $\alpha_l(N,\varkappa)$ we have

$$\alpha_l(N,\varkappa) \longrightarrow \infty \quad \text{at} \ \varkappa\to 1$$

$$(3.10)$$

$$\alpha_l(N,\varkappa) \sim \alpha_l(N) \quad \text{at} \ N \gg 1 .$$

Postulating invariability of the Euclidean metrics in the points not involved in the interpolation, and demanding the continuity of the length $L_d^{cs}$ relative to the dimension d, taking into account the correspondence principle, and using (3.3), (3.6)-(3.9), we obtain:

$$L_d^{cs} = (e-1)/[\exp(1/d) - 1] \qquad (3.11)$$

Let us now turn to the continuous Cantor curve (Fig.2). In correspondence with (2.22), (3.1) its length is equal to

$$L_d^{cc} = \min \lim_{N_k\to\infty} \exp(-\varkappa \ln N_k)\cdot \sum_{a=1}^{N_k} W_a(N_k, \varkappa) \qquad (3.12)$$

where

$$N_k = 4^k$$

$$W_a(N_k,\varkappa) = (g^a_{11}\cos^2\varphi_a + g^a_{22}\sin^2\varphi_a + 2g^a_{12}\sin\varphi_a \cos\varphi_a)^{1/2} \qquad (3.13)$$

$$g^{a}_{\alpha\beta} = G^{k}_{ij}(u^{m}_{k}(x^{1}_{ka},x^{2}_{ka}))\cdot(\partial u^{i}_{k}/\partial x^{\alpha}_{ka})\cdot(\partial u^{j}_{k}/\partial x^{\beta}_{ka}) \qquad (3.14)$$

$$\varkappa = 1/d = 0.5 - 0.5\ln(1-q)/\ln 2$$

$\varphi_a$ is the angle of inclination of the a-th part of the broken line, k is the interpolation number. It should be noted that the finiteness of $L^{cc}_{d}$ as well as (3.4) do not allow, however, to write $W_a(N_k,\varkappa)$ similar to (3.6), since as it is seen from (3.13), the value of $W_a(N_k,\varkappa)$ has not a limit in the points 0 and 1 at $N_k\to\infty$ owing to the angle $\varphi_a$ in it, whereas in (3.6) the existence of such a limit is implied. If $\alpha=r\pi$, where r is a positive rational number, this difficulty may be overcome in the following way. For simplicity, let us limit ourselves to the value of $\alpha=\pi/n$ with $n>2$, there being only n different angles at which the sections of the broken line are disposed. Therefore,

$$N_k = \sum_{i=0}^{n-1} \bar{N}_i \ , \qquad (3.15)$$

where $\bar{N}_i$ is the number of sections of the broken line with inclination angles $\bar{\varphi}_i=i\alpha$, $i=0,1,\ldots,n-1$. Taking into account that $\bar{N}_s\sim\bar{N}_0$, $1\le s\le n-1$ at $k\gg1$, for the length of the Cantor curve we have:

$$L^{cc}_{d} = \exp(-\varkappa\ln n)\cdot\min \lim_{\bar{N}_0\to\infty} \exp(-\varkappa \ln\bar{N}_0)\cdot \sum_{j=1}^{\bar{N}_0} \bar{W}_j(\bar{N}_0,\varkappa) \qquad (3.16)$$

where

$$\bar{W}_j(\bar{N}_0,\varkappa) = \sum_{s=0}^{n-1} W^{s}_{j}(\bar{N}_0,\varkappa) \qquad (3.17)$$

$$W^{s}_{j}(\bar{N}_0,\varkappa) = (\bar{g}^{s}_{11j}\cos^2\bar{\varphi}_s+\bar{g}^{s}_{22j}\sin^2\bar{\varphi}_s+2\bar{g}^{s}_{12j}\sin\bar{\varphi}_s\cos\bar{\varphi}_s)^{1/2} \qquad (3.18)$$

$$0\le s\le n-1, \quad j=1,\ \ldots\ ,\bar{N}_0$$



$\bar{g}^{s}_{\alpha\beta j}$ corresponds to the expression (3.14) taken on the j-th section of the broken line disposed at $\bar{\varphi}_s$. The right hand side of (3.16) to within the factor $\exp(-\varkappa\ln n)$ has the same form as (3.2). Therefore, one may use the same procedure used when deducing (3.11). In result we obtain the following value for the length $L^{cc}_{d}$

$$L^{cc}_{d} = n^{(1-(1/d))}\cdot(e-1)/[\exp(1/d)-1] \qquad (3.19)$$

Further, let us consider the fractals, the measures of which are areas. Since in the infinite interpolation limit the space around the fractal is assumed to become Riemannian and infinite-dimensional, its area is defined according to Ref.[5] as

$$S_d=\min \lim_{N_k\to\infty}(1/2)\sum_{a=1}^{N_k}\int\left(\begin{vmatrix} G^{k}_{i_1j_1} & G^{k}_{i_1j_2}\\ G^{k}_{i_2j_1} & G^{k}_{i_2j_2}\end{vmatrix}|du^{i_1}\wedge du^{i_2}|\cdot|du^{j_1}\wedge du^{j_2}|\right)^{1/2} \qquad (3.20)$$

where $G^{k}_{ij}=G^{k}_{ij}(u^{m}_{k}(x^{1}_{k},x^{2}_{k}))$; $\wedge$ denotes external product; $E^{k}_{a}$ is the a-th element from the minimal cover of the fractal representing a smooth surface with a minimal area in the infinite-dimensional Riemannian space. Fig.4 shows one of the afore-mentioned fractals. It is obtained by removing two mutually perpendicular strips of a width equal to the q-th part of the side of the squares got on the previous step of the interpolation. Let us call this fractal the Cantor square, since it is constructed analogously to the one-dimensional Cantor set (Fig.1). It is noteworthy, that its dimension is equal to the dimension of the Cantor curves (Figs.2,3)

$$d = 2\ln 2/[\ln 2 - \ln(1 - q)] \,. \qquad (3.21)$$

In correspondence with (3.20) the area of the Cantor square is

$$S_d^{cs} = \min \lim_{M_k\to\infty} (1/2) \sum_{a_1=1}^{M_k} \sum_{a_2=1}^{M_k} F_{a_1a_2}(M_k,\varkappa) \exp(-2\varkappa \ln M_k) \qquad (3.22)$$

where

$$M_k = 2^k \,, \qquad \varkappa = 2/d$$

$$F_{a_1a_2}(M_k,\varkappa) = \left( \begin{vmatrix} G^k_{i_1j_1} & G^k_{i_1j_2} \\ G^k_{i_2j_1} & G^k_{i_2j_2} \end{vmatrix} \cdot \begin{vmatrix} \partial u_k^{i_1}/\partial x_k^1 & \partial u_k^{i_2}/\partial x_k^1 \\ \partial u_k^{i_1}/\partial x_k^2 & \partial u_k^{i_2}/\partial x_k^2 \end{vmatrix} \times \begin{vmatrix} \partial u_k^{j_1}/\partial x_k^1 & \partial u_k^{j_2}/\partial x_k^1 \\ \partial u_k^{j_1}/\partial x_k^2 & \partial u_k^{j_2}/\partial x_k^2 \end{vmatrix} \right)^{1/2} \Bigg|_{\substack{x_k^1 = \bar{x}^1_{ka_1a_2} \\ x_k^2 = \bar{x}^2_{ka_1a_2}}} \qquad (3.23)$$

$(\bar{x}^1_{ka_1a_2}, \bar{x}^2_{ka_1a_2})$ are the coordinates of a middle point of the square (on the k-th step of interpolation) of the number $(a_1a_2)$. As in case of the fractals shown in Figs.1 and 2, the demand of finiteness of $S_d^{cs}$ with due regard for (3.4) and correspondence principle at $\varkappa\to 1$ results in the following expression for $F_{a_1a_2}(M_k,\varkappa)$:

$$F_{a_1a_2}(M_k,\varkappa) = [H_{a_1}(M_k,\varkappa)H_{a_2}(M_k,\varkappa) - D_{a_1a_2}(M_k,\varkappa)]\Phi(M_k,\varkappa) \qquad (3.24)$$

where the function $H_a(M_k,\varkappa)$ has the same form as in (3.7), and



$$S_d^{cs} = \min \lim_{M_k\to\infty} (1/2)\, \Phi(M_k,\varkappa) \qquad (3.25)$$

$D_{a_1a_2}(M_k,\varkappa)$ is a sufficiently arbitrary function of a structure analogous to (3.8). Then using the same method used when deducing the values of the lengths (3.11) and (3.19), we get

$$S_d^{cs} = [(e-1)/(\exp(2/d)-1)]^2 \qquad (3.26)$$

And finally, let us calculate the area of the fractal (Fig.5), which is in a sense the generalization of Serpinsky's carpet [1,2]. It is constructed in such a way, that at each k-th step of interpolation one removes the central square out of $(2n+1)^2$ ones obtained by dividing every square got at the previous step. The dimension of the given fractal is

$$d = \ln(4n(n+1))/\ln(2n+1) \qquad (3.27)$$

Since in contrast to the previous fractals the points not involved in the interpolation are not isolated points but are the four sides and vertices of the unit square, the origin of coordinates for the convenient arithmetization of points is chosen on the centre of the square (Fig.5), at each k-th step of interpolation there being assumed the following numbering system: $a_1=1$ corresponds to the first row of the squares gathering around the square that is removed first (at k=1), $a_1=2$ corresponds to the second row gathering around the first one, and so on; $a_2$ numbers the squares in each row. It follows from (3.20) that the area of the generalized Serpinsky's carpet (Fig.5) is

$$S_d^{sc} = \min \lim_{k\to\infty} (1/2) \sum_{a_1=1}^{M_k} \sum_{a_2=1}^{N_k(a_1)} F_{a_1a_2}(M_k,\varkappa)\, (1/(2n+1))^2 \qquad (3.28)$$

where

$$M_k = n(2n+1)^{k-1}, \qquad \varkappa = 1/d$$

$F_{a_1 a_2}(M_k,\varkappa)$ has the same form as in (3.23), $N_k(a_1)$ is the number of squares in the row with number $a_1$. Denoting

$$\Phi_{a_1}(M_k,\varkappa) = \sum_{a_2=1}^{N_k(a_1)} (1/2)\, F_{a_1 a_2}(M_k,\varkappa), \qquad (3.29)$$

one is able to show that the requirement of finiteness of $S_d^{sc}$ results in the following general form of $\Phi_{a_1}(M_k,\varkappa)$:

$$\Phi_{a_1}(M_k,\varkappa) = [2a_1 + D_{a_1}(M_k,\varkappa)]((2n+1)/n)^2\, S_n^k \qquad (3.30)$$

where

$$\min \lim_{k\to\infty} S_n^k = S_d^{sc} \qquad (3.31)$$

$D_{a_1}(M_k,\varkappa)$ is a sufficiently arbitrary function for which

$$\lim_{k\to\infty} \sum_{a_1=1}^{M_k} D_{a_1}(M_k,\varkappa)\cdot M_k^{-2} = 0 .$$

Then using (3.28)-(3.31) and being guided by the same reasoning that when obtaining (3.11), (3.19) and (3.26), we have

$$S_d^{sc} = 2n/(2n+1) \qquad (3.32)$$



## 4. CONCLUSION

Summarizing the aforesaid, let us enumerate all assumptions used in calculation of concrete measures of fractals:

(i) The space around the fractal is infinite-dimensional and Riemannian.

(ii) In the points not involved in the interpolation the Euclidean metrics is not changed.

(iii) There is the dependence (2.22), where $g_k^{m}(x_k^1,\ldots,x_k^n)$ are sufficiently partly-smooth functions.

(iv) The dimension of the fractal is determined by demanding the finiteness of the d-volume of the fractal (2.20).

(v) The measure of the fractal defined as the limit of the measure of the corresponding prefractal object is assumed to be finite and not equal in general to zero or infinity.

(vi) All geometrical values (metrics, measure, etc.) are continuous relative to the fractal building parameter.

(vii) The lines, surfaces, etc., connecting correspondingly the fixed points, lines, etc., in the infinite-dimensional Riemannian space corresponding to straight lines, planes, etc. in the primordial Euclidean space (in partly-linear prefractal objects) have minimal measures (e.g., lines are geodesics, and so on).

It should be noted that postulation of the dependence (2.22) and properties (2.23)-(2.28) are necessary for "sewing together" the two geometries - the usual Euclidean geometry of partly-smooth objects, and the fractal's eigengeometry.

Figs.6,7 show this connection qualitatively in the form of one of possible dependences of the lengths of prefractal objects on an interpolation number for a Cantor set (Fig.1) and a continuous Cantor curve (Fig.2), respectively.

It is seen from the figures that at $k \geq k_0$ there occurs an essential change in the geometry of the space around the fractals. Since at present the metric tensor $G^k_{ij}(u)$ cannot be determined uniquely for every particular fractal, the behaviour of $L_k$ lengths in Figs.6,7 has only a preliminary and qualitative character. But the existence of $k_0$, nearby which the geometries are "sewn together", is necessary.

Then let us present for comparison the expressions of lengths for the continuous ($L^{cc}_d$) and ruptured ($L^{dc}_d$) Cantor curves (Figs.2,3), respectively

$$L^{cc}_d = n^{(1-(1/d))}(e-1)/[\exp(1/d)-1], \qquad (4.1)$$

$$L^{dc}_d = 2^{(1-(1/d))}(e-1)/[\exp(1/d)-1] , \qquad (4.2)$$

where $n=\pi/\alpha>2$; $\alpha$ is the angle shown in Fig.2; d is the dimension which is the same for both of the fractals and is equal to

$$d = 2\ln 2/[\ln 2 - \ln(1-q)] , \qquad (4.3)$$

It follows from (4.1)-(4.3) that the only geometrical characteristic distinguishing between the fractals in Figs.2,3 is the length determined in the new geometry according to (3.1) (in the former Euclidean geometry it is equal to infinity for both of them).

In conclusion also note, that the given geometry allows to describe uniformly both "rational" geometrical objects, such as usual partly-smooth geometrical figures obtained by a finite



number of operations and "irrational", "transcendental" objects formed as a result of an infinite series of operations. As to the unique determination of the metrics of the space corresponding to the fractal, it is apparently coupled with theory of information and requires a rather thorough study.

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## APPENDIX

In correspondence with (2.6)-(2.10), the functions

$$w_m(u) = \exp[-(u^m/\lambda_m)^2 \cdot \varphi_m(u^m)/(1+\tau_m^2(u^m)^2)], \quad m\geq 1 \qquad (A.1)$$

$$\Lambda_m = \lambda_m \sqrt{\pi} \qquad (A.2)$$

$$\rho_m = 1 - \exp[-c_m^2(d,\Delta)/\lambda_m^2] \qquad (A.3)$$

can serve as $\omega_m(u)$, $\Lambda_m$ and $\rho_m$, where

$$\lambda_m^2 = (f_m(\Gamma^2(-d,\Delta))/f_m(\Gamma^4(d-m+1,\Delta)))\cdot l_m(d,\Delta) , \qquad (A.4)$$

$$\tau_m^2 = (f_m(\Gamma^4(d-m+1,\Delta))/f_m(\Gamma^4(-d,\Delta))) \cdot t_m(d,\Delta) . \qquad (A.5)$$

The functions $\varphi_m(u^m)$ have the following properties:

$$0 < \varphi_m(u^m) \leq 1 , \quad \varphi_m(0) = 1 \qquad (A.6)$$

and furthermore, they have limited derivatives and are equal to zero outside the region occupied by the fractal; $\Delta=|d-d_0|$; $\Gamma(x,\Delta)$ is an additional incomplete $\gamma$-function; $f_m(x)$ are positive whole functions, e.g., $\exp(\beta_m x)$ with $\beta>0$; $l_m(d,\Delta)$, $t_m(d,\Delta)$ are functions not equal to 0 or $\infty$ at the limit $d\to d_0$ for any fixed $m\geq 1$, and the choice of the dependence $\tau_m$ on $d$ and $d-d_0$ in the form of (A.5) is conditioned by the following consideration: $c_m^2(d,\Delta)$ are sufficiently arbitrary functions for which the following relation occurs:

$$\sum_{m=1}^{\infty} \exp[-c_m^2(d,\Delta)/\lambda_m^2(d,\Delta)] = d \qquad (A.7)$$

Let us find constraints on the functions $f_m(x)$, $l_m(d,\Delta)$,



$t_m(d,\Delta)$ by the condition (2.17). Let us introduce the signs

$$s = \Gamma^{-2}(-d,\Delta)$$

$$a_m(s^{-1}) = f_m(\Gamma^4(-d,\Delta))/f_m(\Gamma^2(-d,\Delta))$$

$$b_m(s) = [l_m(d,\Delta)\cdot t_m(d,\Delta)]^{-1} , \quad m \geq 1 \qquad (A.8)$$

$$\Omega(u) = \prod_{n=1}^{\infty} \omega_n(u)$$

Expanding the functions $a_m(s^{-1})$ and $b_m(s)$ into a series correspondingly over $s^{-1}$ and $s$

$$a_m(s^{-1}) = \sum_{n=0}^{\infty} a_{mn} s^{-n} ,$$

$$b_m(s) = \sum_{n=0}^{\infty} b_{mn} s^{n} , \qquad (A.9)$$

taking into account (A.1), and demanding that the matrix

$$h_{m,l} = \sum_{k=1}^{\infty} a_{mk}^T b_{kl} ; \quad m,l \geq 0 \qquad (A.10)$$

should satisfy the conditions

$$h_{m,m+j} = 0 , \quad -m \leq j < 0 , \qquad (A.11)$$

$$\sum_{m=0}^{\infty} h_{m,m} < \infty , \qquad (A.12)$$

one can secure that

$$\lim_{d\to d_0} \Omega(u) \neq 0, \infty .$$

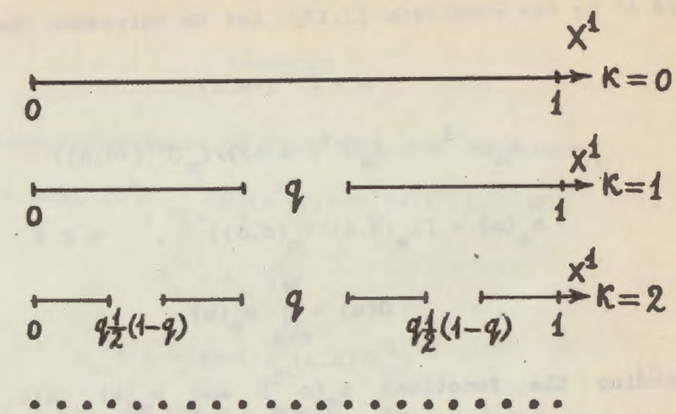


Fig.1

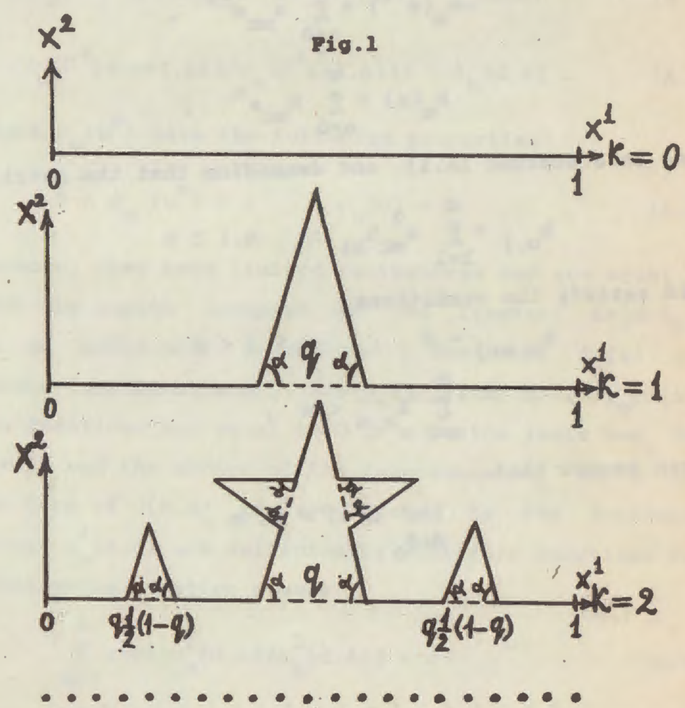


Fig.2



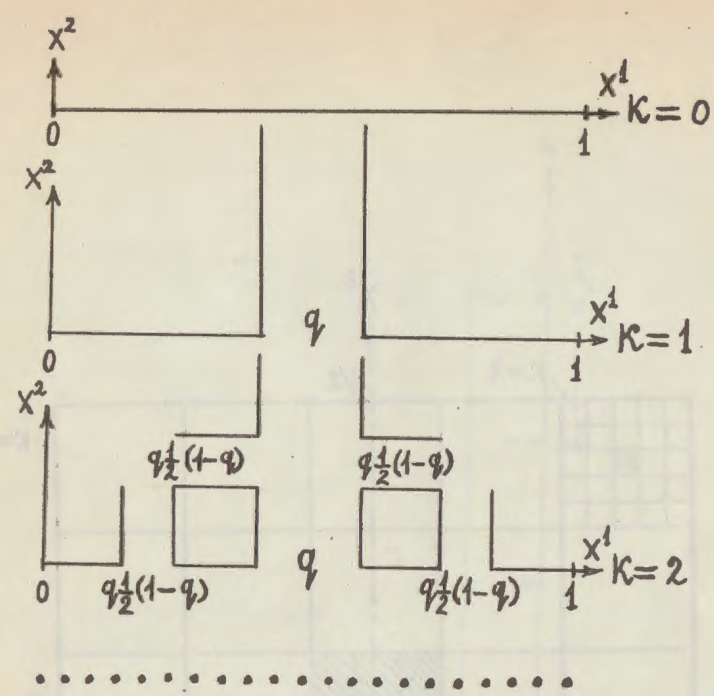


Fig.3

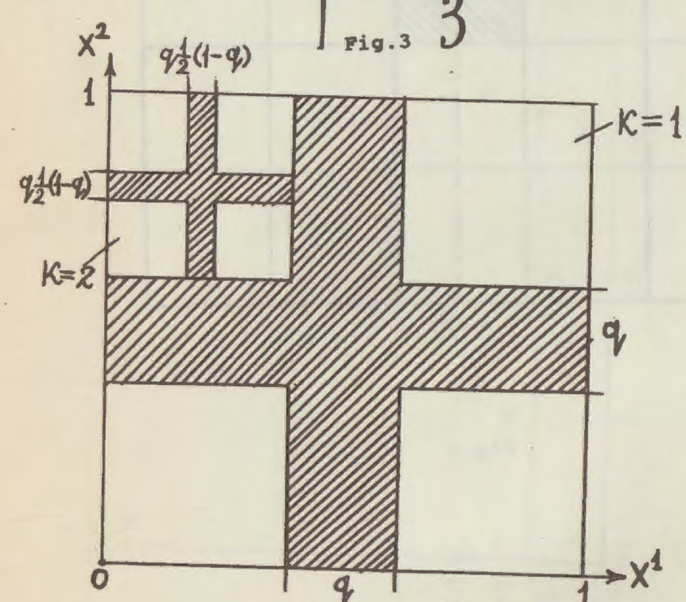


Fig.4

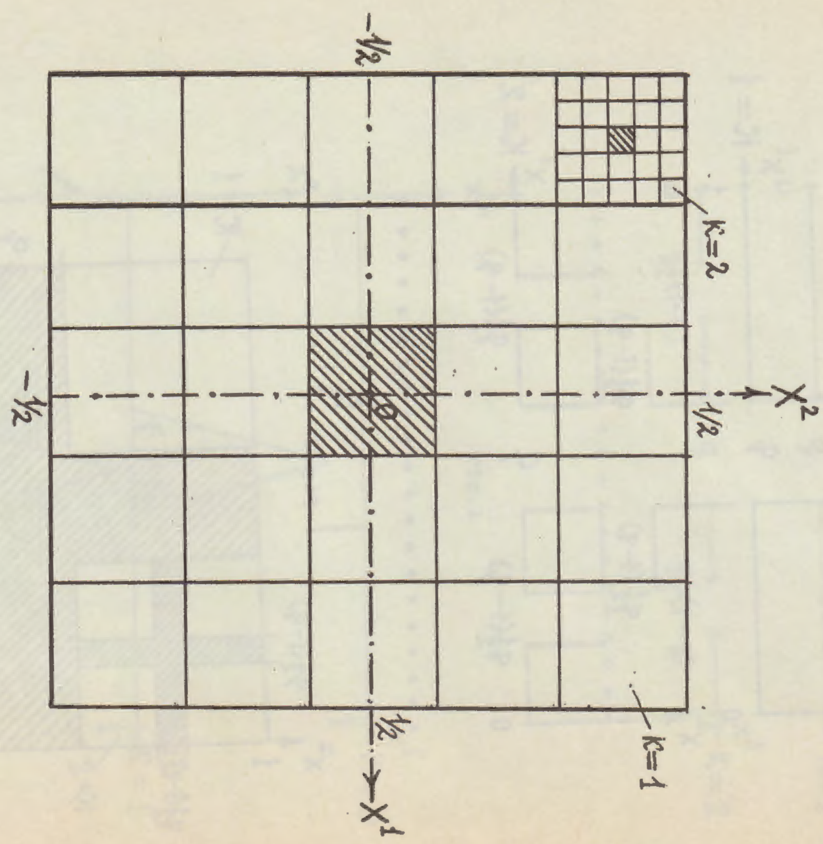


Fig.5



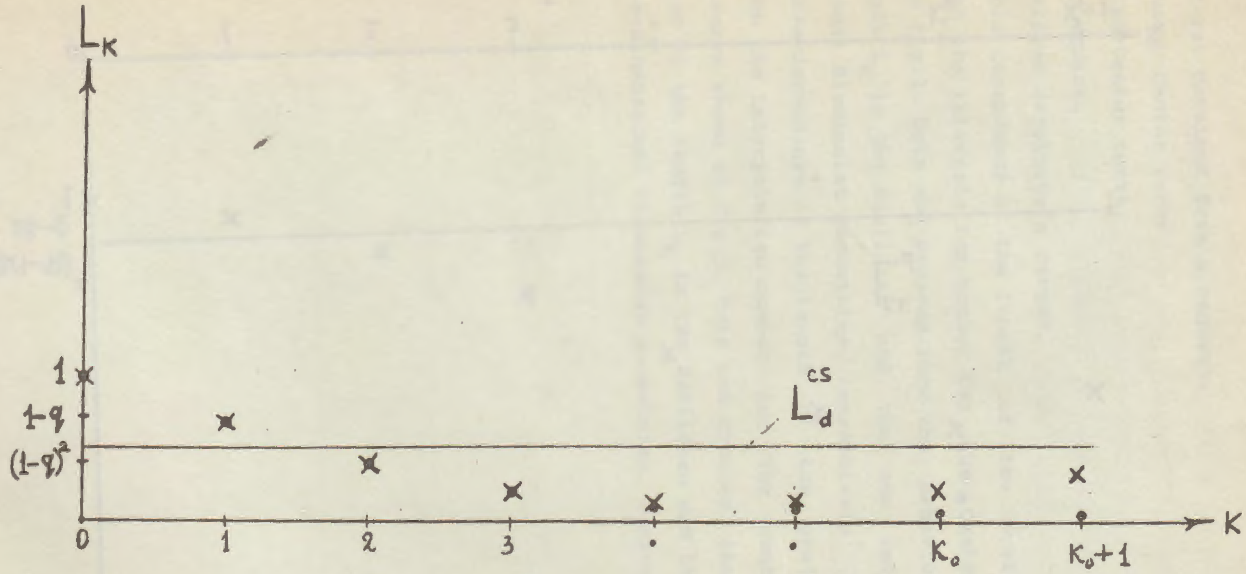


Fig.6

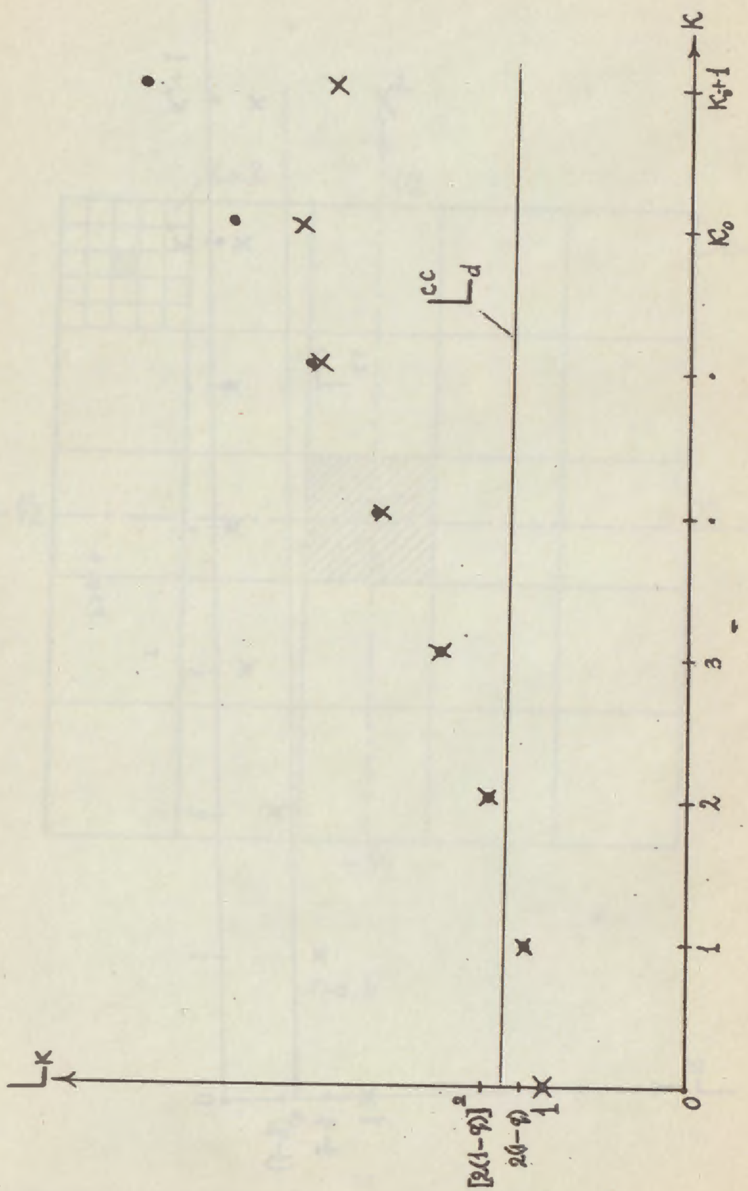


Fig.7



Figure Captions

Fig.1 A Cantor set obtained from a segment.

Fig.2 A continuous Cantor curve.

Fig.3 A ruptured Cantor curve.

Fig.4 A Cantor square.

Fig.5 A generalized Serpinsky's carpet.

Fig.6 A possible dependence of the length of the prefractal object on the interpolation number for the Cantor set shown in Fig.1. Dots and crosses show the behaviour of the length $L_k$ in the Euclidean and the new infinite-dimensional Riemannian geometries, respectively.

Fig.7 A possible dependence of the length of the prefractal object on the interpolation number for the continuous Cantor curve shown in Fig.2. Dots and crosses show the behaviour of the length $L_k$ in the Euclidean and the new infinite-dimensional Riemannian geometries, respectively.

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