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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
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OPTIMAL CODES FOR SHANNON NOISY CHANNELS AND RANDOM ENERGY MODEL FOR ARBITRARY NUMBER OF COLORS

ЦНИИатоминформ
ЕРЕВАН-1990

Նախնատիպ ԵրՖԻ-1296(82)-90

Դ.Բ.ՍԱՀԱԿՅԱՆ

ԱՂՄՈՒԿՈՎ ՈՒՂԻՆԵՐԻ ՀԱՄԱՐ ԼԱՎԱԳՈՒՅՆ ԿՈԴԵՐ ԸՍՏ ՇԵՆՈՆԻ, ԵՎ ՊԱՏԱՀԱԿԱՆ ԷՆԵՐԳԻԱՆԵՐԻ ՄՈԴԵԼԸ ԳՈՒՅՆԵՐԻ ԿԱՄԱՅԱԿԱՆ ԹՎԻ ԴԵՊՔՈՒՄ

Դիտարկվում է սպինային ապակու Դերիդայի մոդելը Z(Q) համաչափության խմբով կամայական Q-ի համար։ Ցույց է տրված, որ այդ մոդելով կոդավորումը տալիս է ինֆորմացիայի հաղորդման առավելագույն արագությունը, որը հնարավոր է աղմուկով ուղիների համար ըստ Շենոնի թեորեմի։

Երևանի ֆիզիկայի ինստիտուտ
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Information transmission is always accompanied with noise-induced errors. In order to derive information with a minimum error probability, one has to insert extra bits of information to the transmitted message.

For the transmission rate τ (bit/sec) in a band of width W, signal energy E and noise energy N_0 Shannon obtained a limit

$$\tau \leq \frac{W}{2} \log_2 (1 + 2E/N_0) . \qquad (1)$$

For the case $E \ll N$, from (1) one can obtain [1] a limit for the transmission rate (ratio of the number of useful bits to their total number):

$$\tau \leq (E/N_0)/\ln 2 . \qquad (2)$$

The codes used in practice do not saturate limits (1), (2).

N.Sourlas in his remarkable paper [1] pointed out that the random energy model with two colors [2,3] saturates the limit (2) for the binary alphabet.

In the present paper we will prove that limit (2) actually is achieved for the alphabet with an arbitrary number of colors Q . Here we will use the solution of the random energy model for the symmetry group $Z(Q)$ with vector spin interaction [4] .

Consider the hamiltonian

$$H = \sum_{1\leq i_1 < i_2 \dots i_p} \mathrm{Re}\,(j^{0}_{i_1 \dots i_p} + j^{1}_{i_1 \dots i_p}) \sigma_{i_1} \cdots \sigma_{i_p} . \qquad (3)$$

Here σ_i take the values $\exp(i2\pi K/Q)$, $K = 1, Q$. Let we want to remember the spin state

$$\sigma_i = S_i$$

Then at

$$j^{0}_{i_1 \dots i_p} = J_0 S^{*}_{i_1} \cdots S^{*}_{i_p} \frac{p!}{N^{p-1}} \qquad (4)$$

we obtain that hamiltonian (3) with constants (4) has as its ground state $\sigma_i = S_i$ with energy $J_0 N$. This is a ferromagnetic phase.

Switch on nonzero $j^{1}_{i_1 \dots i_p}$ with Gaussian distribution

$$\langle (j^{1}_{i_1 \dots i_p})^2 \rangle = \frac{p!}{N^{p-1}} J^2 . \qquad (5)$$

Then, up to some critical value J/J_0 the total magnetization of the system will retain which then will vanish completely. Via the gauge transformation Eq.(3) can be reduced to the form:

$$H = \sum \mathrm{Re}\,(j^{0} + j^{1}_{i_1 \dots i_p}) \sigma_{i_1} \dots \sigma_{i_p} \qquad (6$$

$$\langle Z^n \rangle = \prod_{a<1} \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ_{ab} \int_{-i\infty}^{i\infty} \frac{1}{2\pi} d\lambda_{ab} \prod_{a=1}^{n} \int_{-\infty}^{\infty} dS_a \int_{-i\infty}^{i\infty} dt_{ab} \, e^{-NG(Q_{ab}, \lambda_{ab}, S_a, t_a)}$$

Using the replica method in the mean field approximation (exact in our case) we obtain

$$-G = \frac{B^2 J^2}{4} (n + \sum_{a\neq b} Q^{p}_{ab}) + BJ_0 \sum_a S^{p}_a - \sum_{a\neq b} \frac{Q_{ab}\lambda_{ab}}{2} - \sum_a S_a t_a + \qquad (7)$$
$$+ \ln Tr \exp \left[\frac{1}{2} \sum_{a\neq b} \lambda_{ab} \sigma_a \sigma_b + \sum_a \sigma_a t_a \right]$$

We will search for solution of (8) from the extremum condition (7) in the case of replica symmetry conservation

$$\frac{-BF}{N} = \frac{B^2 J^2}{4} (1 - q^p) + \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\infty} e^{-\frac{z^2}{2}} z dz \ln \sum_{K=1}^{Q} \exp\left(\sqrt{\lambda} z \cos(\varphi - \varphi_K) + t \cos\varphi_K \right) +$$
$$+ BJ_0 S^p - St + \frac{(Q-1)}{2} \qquad (8)$$

Here $\varphi_K = 2\pi/Q$.

At the extremum point

$$\frac{B^2 J^2}{2} p q^{p-1} = 2 \qquad (9)$$

$$BJ_0 p S^{p-1} = t \qquad (10)$$

At $q = 1$ we have

$$\lambda \to \infty , \quad t \to \infty \Rightarrow \qquad (11)$$

$$-BF = BJ_0 \qquad (12)$$

At $q < 1$ $\lambda \to 0$ according to (9). At $\lambda = 0$ we obtain

$$S = \frac{\sum_K \cos\varphi_K \exp(t\cos\varphi_K)}{\sum_K \exp(t\cos\varphi_K)} \qquad (13)$$

$$Q = S^2 \tag{14}$$

Finally, instead of (11), (12) we have

$$t=0, \quad q=0, \quad s=0 \tag{15}$$

$$-\frac{BF}{N} = \frac{B^2 J^2}{4} + \ln Q \tag{16}$$

Now we consider the case of replica symmetry breaking. Let n decomposes into n/m subgroups with m copies in each. For such a one-step scheme of replica symmetry breaking we have

$$-\frac{BF}{N} = \frac{B^2J^2}{4} + \left[(m-1)q_1^p - mq_o^p\right]\frac{B^2J^2}{4} - \frac{1}{2}\left[(m-1)\;q_1 - m\lambda_o q_o\right]$$

$$-\frac{\lambda_1}{2} + \frac{1}{m2\pi}\int_0^{2\pi} d\varphi_o \int_0^{\infty} z_o dz_o \exp\left(-\frac{z_o^2}{2}\right) \ln \frac{1}{2\pi}\int_0^{2\pi} d\varphi_1 \int_0^{\infty} z_1 dz_1 \times$$

$$\times \exp(-z_1^2/2)\Big[\sum_K \exp\big(\sqrt{\lambda_1-\lambda_o}\; z_1 \cos(\varphi_1-\varphi_K) +$$

$$t\cos\varphi_K + \sqrt{\lambda_o}\; z_o\cos(\varphi_o-\varphi_K)\big)\Big]^m$$

The case $q_o < q_1$ is most interesting. This takes place at

$$q_o=0, \quad q_1=1, \quad \lambda \to \infty, \quad s=0, \quad t=0 \tag{17}$$

$$-\frac{BF}{N} = \frac{BB_c}{2} \tag{18}$$

$$B_c = 4J^2 \ln Q \tag{19}$$

We are interested in system behavior at $B \to \infty$.

At $J_o > \frac{B_c}{2} \equiv J\ln Q$ we have

$$-\frac{BF}{N} = \frac{BB_c}{2} \tag{20}$$

$$S = 1 \qquad \text{(total magnetization)} \tag{21}$$

At $J_o < J\sqrt{\ln Q}$

$$-\frac{BF}{N} = \frac{BB_c}{2} \tag{22}$$

$$S = 0 \qquad \text{(no magnetization)} \tag{23}$$

By ideology of [1] , Eq.(21) corresponds to decoding without errors (the error probability is zero), and Eq.(23) - to that with errors:

$$\tau = N/(P^p/P!) = P!/N^{p-1} \tag{24}$$

On the other hand,

$$\frac{\langle J^2 \rangle}{J_o^2} = \frac{P!}{N^{p-1}} \ln Q \tag{25}$$

Thus we arrive at limit (2):

$$\tau = \frac{E}{N_o}\,\frac{1}{\ln Q}$$

It would be interesting to obtain limit (1) for the case $E/N_o \sim 1$ using model (3) with rare couplings.

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Д.Б.СААКЯН

ОПТИМАЛЬНЫЕ КОДЫ ДЛЯ КАНАЛОВ С ШУМОМ ПО ШЕННОНУ И МОДЕЛЬ СЛУЧАЙНЫХ ЭНЕРГИЙ ДЛЯ ПРОИЗВОЛЬНОГО ЗНАЧЕНИЯ ЧИСЛА ЦВЕТОВ

(на английском языке, перевод З.Н.Асланян)

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