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Preprint YERPHI-1308(9)-93

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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE

YERPHI-1308-9-93

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# PSEUDOCLASSICAL FOLDY-WOUTHUYSEN TRANSFORMATION AND THE CANONICAL QUANTIZATION OF THE D=2N DIMENSIONAL SPINNING PARTICLE IN THE EXTERNAL ELECTROMAGNETIC FIELD

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# Pseudoclassical Foldy-Wouthuysen transformation and the canonical quantization of the D=2n dimensional spinning particle in the external electromagnetic field

The canonical quantization of D=2n dimensional Dirac spinning particle in the external electromagnetic field is carried out in the gauge which allows to describe simultaneously particles and antiparticles (massive and massless) already at the classical level. Pseudoclassical Foldy-Wouthuysen transformation is used to obtain canonical (Newton-Wigner) coordinates and in terms of these variablesthe theory is quantized. The connection of this quantization with the Blount picture of Dirac particle in the external electromagnetic field is discussed.

Yerevan Physics Institute
Yerevan 1993

## 1 Introduction

In papers [1-3] the relativistic spinning particle was canonically quantized in the free case (in $D = 2n$ dimensions) and in the external electromagnetic field ($D = 4$). The pseudoclassical description of the particle (see [4-7]) was used in these papers and the quantization scheme was characterized by introduction at the classical level of all gauge fixing constraints. This quantization was shown to result in the Dirac theory in the Foldy-Wouthuysen representation in the free particle case (see also [8] ), while for the spinning particle in the external electromagnetic field a generalization of the Blount's picture [9] was obtained. The quantization was carried out not in the terms of the initial variables of the theory, but in the terms of new (Newton-Wigner ) variables, for which the quantum commutation relations are canonical. The transition to canonical variables prior to quantization seems more appropriate, considering that in the case of the relativistic particle in the external electromagnetic field the operator realisation of the theory in terms of the initial variables appears impossible due to the complexity of corresponding Dirac brackets.

In this paper a method of constructing of the Newton-Wigner type variables at the classical level using the pseudoclassical analog of the Foldy-Wouthuysen transformation is proposed. This allows to obtain straightforwardly the relation between the canonical and initial variables bypassing the calculation of the Dirac brackets with the subsequent diagonalization of these brackets. The investigation is carried out in the space-time dimensions $D = 2n$. Note that pseudoclassical Foldy-Wouthuysen transformation was used in [10] to obtain the expression for the hamiltonian of the electron interacting with the external constant magnetic field, which after quantization automatically brought to diagonalized hamiltonian.

## 2 Constraints

Consider the action of the theory, describing the relativistic spinning particle in the external electromagnetic field in D=2n [4],[6], [11]

$$L = \frac{1}{2}\int d\tau [\frac{(\dot{x}^\mu)^2}{e} + em^2 - i\left(\xi_\mu \dot{\xi}^\mu - \xi_{D+1}\dot{\xi}_{D+1}\right) - i\chi\left(\frac{\xi_\mu \dot{x}^\mu}{e} - m\xi_{D+1}\right) + \\ +2g\dot{x}^\mu A_\mu + igeF_{\mu\nu}\xi^\mu\xi^\nu], \tag{1}$$

here $x^\mu$ is particle coordinate, $\mu = 0,1,2,\ldots,D-1$, $\xi^\mu$ is Grassmann variables, describing spin degrees of freedom, $\xi_{D+1}, \chi$ and $e$ are additional fields (e is an even element, $\xi_{D+1}$ and $\chi$ are odd elements of Grassmann algebra) , $g$ is the charge of the particle, $A^\mu$ is the vector-potential of the electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , the overdote denotes the differentiation over $\tau$ along the trajectory; the derivatives over Grassmann variables are left. Following the steps analogous to those of [3] we come to the complete set of constraints

$$\Phi_\mu = \pi_\mu - \frac{i}{2}\xi_\mu, \mu = 0,1,2,\ldots,D-1; \quad \Phi_D = \pi_{D+1} + \frac{i}{2}\xi_{D+1}, \tag{2}$$

$$\Phi_{D+3} = \mathcal{P}^2 - m^2 - igF_{\mu\nu}\xi^\mu\xi^\nu, \quad \Phi_{D+4} = x'_O, \quad \Phi_{D+5} = \pi_e, \quad \Phi_{D+6} = e + \frac{1}{\tilde{\omega}}, \tag{3}$$

$$\Phi_{D+1} = \mathcal{P}_\mu\xi^\mu - m\xi_{D+1}, \quad \Phi_{D+2} = a\xi_O + b\xi_{D+1}, \tag{4}$$

here $\pi_\mu, \pi_{D+1}, \pi_e$ are momenta, canonically conjugate to $\xi^\mu, \xi_{D+1}$ and $e$ - respectively, $\mathcal{P}_\mu = P_\mu - gA_\mu, \mathcal{P}_O = -\kappa\tilde{\omega}, \tilde{\omega} = \sqrt{\mathcal{P}_i{}^2 + m^2 + igF_{\mu\nu}\xi^\mu\xi^\nu}$. Recall [3], that $x'_O = x_O - \kappa\tau$ and the constraint $\Phi_{D+4} = x_O - \kappa\tau$ , which is one of the gauge fixing constraints, was transformed into Eq.(3) by a canonical transformation of variables $x^\mu, P_\mu$ to $x'^\mu, P'_\mu$ defined by the relations

$$x'_O = x_O - \kappa\tau, \quad x'^i = x^i, \quad P'_\mu = P_\mu \tag{5}$$

(the corresponding generating function is $W = x^\mu P'_\mu - \tau\kappa P'_O$ ). The value $\kappa = +1$ corresponds to a particle, $\kappa = -1$ corresponds to a antiparticle. For the independent variables of the theory we choose $x^i, \mathcal{P}_i, \xi^i$.

## 3 Foldy-Wouthuysen Transformation

Let us now consider a pseudoclassical canonical Foldy - Wouthuysen transformation with a generator of the infinitesimal canonical transformation [10]

$$S_{\rm d} = -2i(\mathcal{P}_j\xi_j)\xi_{D+1}\theta, \tag{6}$$

where $\theta$ is a function of the variables of the theory which will be specified later. The result of the finite canonical transformation of any dynamical quantity $f$ is given by the expression [12]

$$\tilde{f} = e^{\widehat{S_{\rm d}}} f = f + \{f, S_{\rm d}\}^* + \frac{1}{2!}\{\{f, S_{\rm d}\}, S_{\rm d}\}^* + \ldots, \tag{7}$$

where $\{,\}^*$ denotes the Dirac brackets for the (2).For the variables of the theory we have

$$\{x_\mu, \mathcal{P}_\nu\}^* = g_{\mu\nu}, \quad \{\xi_\mu, \xi_\nu\}^* = ig_{\mu\nu}, \quad \{\xi_{D+1}, \xi_{D+1}\}^* = -i, \quad \{\mathcal{P}_\mu, \mathcal{P}_\nu\}^* = gF_{\mu\nu} \tag{8}$$

(all other brackets vanish).

Applying Eq. (7) to a function A of the independent variables $x^i, \mathcal{P}_i, \xi^i$ and taking into account the relations

$$\begin{aligned} &\{A\xi_{D+1}, S_{\rm d}\}^* = A(2\theta)(\mathcal{P}_j\xi_j), \\ &\{A(\mathcal{P}_j\xi_j), S_{\rm d}\}^* = -(2\theta)\gamma A\xi_{D+1} + (\mathcal{P}_j\xi_j)\xi_{D+1}R_1, \\ &\{A(\mathcal{P}_j\xi_j)\xi_{D+1}, S_{\rm d}\}^* = 0, \end{aligned} \tag{9}$$

where $\gamma = i\{(\mathcal{P}_i\xi_i), (\mathcal{P}_j\xi_j)\}^* = \mathcal{P}_i{}^2 + igF_{ij}\xi_i\xi_j$ , $\mathrm{R}_1$ is a function of the variables of the theory, we find for $\tilde{A}$ the expression

$$\begin{aligned} \tilde{A} = A - &\frac{i}{\sqrt{\gamma}}\{A, (\mathcal{P}_j\xi_j)\}^*\xi_{D+1}\sin(2\theta\sqrt{\gamma}) + \\ &+\frac{i}{\gamma}\{A, (\mathcal{P}_j\xi_j)\}^*(\mathcal{P}_k\xi_k)(\cos(2\theta\sqrt{\gamma}) - 1) + (\mathcal{P}_i\xi_i)\xi_{D+1}\mathrm{R}_2, \end{aligned} \tag{10}$$

where $R_2$, like $R_1$, depends on the variables of the theory, the explicit expressions of $R_2$ and $\mathrm{R}_1$ are immaterial. If we now specify the function $\theta$ by taking $tg(2\theta\sqrt{\gamma}) = \frac{\sqrt{\gamma}}{m}$, and hence $\sin(2\theta\sqrt{\gamma}) = \frac{\sqrt{\gamma}}{\omega}$, $\cos(2\theta\sqrt{\gamma}) = \frac{m}{\omega}$, then

$$\tilde{A} = A - i\{A, (\mathcal{P}_i\xi_i)\}^*\frac{\xi_{D+1}(\omega + m) + (\mathcal{P}_j\xi_j)}{\omega(\omega + m)} + (\mathcal{P}_i\xi_i)\xi_{D+1}\mathrm{R}_2, \tag{11}$$

where $\omega = \sqrt{\mathcal{P}_i^2 + m^2 + igF_{ij}\xi_i\xi_j} = \sqrt{\gamma + m^2}$.

## 4 Final Dirac Brackets and Quantisation

The next step will consist of the prove, that the variables $\tilde{x}^i, \tilde{\mathcal{P}}_i, \tilde{\xi}^i$ obtained by application of Eq.(11) to the variables $x^i, \mathcal{P}_i, \xi^i$ are Newton-Wigner variables, i.e. the final Dirac brackets for these variables are canonical. To achieve this end we use the relation

$$\{\tilde{A}, \tilde{B}\}_{D(\Phi)} = \{A', B'\}_{D(\Phi)}, \tag{12}$$

where $A' \equiv \tilde{A}|_{\Phi=0}, \quad B' \equiv \tilde{B}|_{\Phi=0}$, $\{\ldots, \ldots\}_{D(\Phi)}$ denotes the Dirac brackets for the complete set of constraints (2)-(4) . The equation Eq.(12) is a reflection of the property of the Dirac brackets which states that the Dirac brackets of the constraints with any dynamical quantity vanish. From the constraints Eq. (4) we find

$$\xi_{D+1} = -\frac{a(\mathcal{P}_j \xi_j)}{\tilde{\beta}}, \quad \xi_0 = \frac{b(\mathcal{P}_j \xi_j)}{\tilde{\beta}}, \tag{13}$$

where $\tilde{\beta} = -b\kappa\omega + am$. Substituting now Eq. (13) in Eq. (11) we get

$$A' \equiv \tilde{A}|_{\Phi=0} = A + i\{A, (\mathcal{P}_i \xi_i)\}^{*}(\mathcal{P}_j \xi_j)\frac{(b\kappa + a)}{\tilde{\beta}(\omega + m)}. \tag{14}$$

Using Eq. (14) we find for variables $x'^i, \mathcal{P}'_i, \xi'^i$ the expressions

$$\begin{aligned} x'_i &= x_i - i\xi_i(\mathcal{P}_j \xi_j)\frac{(b\kappa + a)}{\tilde{\beta}(\omega + m)} \equiv q_i, \\ \mathcal{P}'_i &= \mathcal{P}_i + igF_{ij}\xi_j(\mathcal{P}_k \xi_k)\frac{(b\kappa + a)}{\tilde{\beta}(\omega + m)} \equiv \pi_i, \\ \xi'_i &= \xi_i + \mathcal{P}_i(\mathcal{P}_j \xi_j)\frac{(b\kappa + a)}{\tilde{\beta}(\omega + m)} \equiv \psi_i. \end{aligned} \tag{15}$$

Consider now the right side of the equation (12). We have by definition

$$\{A', B'\}_{D(\Phi)} = \{A', B'\}^{**} - \{A', \varphi_r\}^{**} C^{-1}_{rr'} \{\varphi_{r'}, B'\}^{**}. \tag{16}$$

Here $\varphi_r = (\Phi_{D+1}, \Phi_{D+2})$, $\{\ldots, \ldots\}^{**}$ stands for the Dirac brackets for a subset of constraints (2),(3), $C^{-1}$ is the inverse matrix of

$$C_{rr'} = \{\varphi_r, \varphi_{r'}\}^{**} \tag{17}$$

Now we'll take an advantage of the special structure of the constraints (3): one of each pair of constraints is a canonical variable . This allows to prove that

$$\{F,G\}^{**} = \{F,G\}^{*} \tag{18}$$

for any dynamical variables $F$ and $G$ (see e.g.[13]). With account of Eq. (18 ) the formula (16 ) takes the form

$$\{A',B'\}_{D(\Phi)} = \{A',B'\}^{*} - \{A',\varphi_r\}^{*}C^{-1}_{rr'}\{\varphi_{r'},B'\}^{*}. \tag{19}$$

The matrix $C$ is given by

$$C_{rr'} = \{\varphi_r,\varphi_{r'}\}^{*} = \begin{pmatrix} O & -i\alpha \\ i\alpha & i(\alpha^2 - b^2) \end{pmatrix}, \tag{20}$$

where $\alpha = -a\kappa\omega + bm$.

If now we consider functions $A', B'$ which depend only on variables $x^i, \mathcal{P}_i, \xi^i$ (e.g. functions in (15)), then taking into account the relations

$$\{A',\xi_O\}^{*} = \{A',\xi_{D+1}\}^{*} = 0, \tag{21}$$

and similar relations for $B'$, it's easy to check that the second summand in Eq. (19) consists of only one term, which contains the matrix element $C^{-1}_{D+1,D+1}$. Substituting in Eq. (19) the expressions for $A'$ and $B'$ from (14) one can show by direct calculations that

$$\{A',B'\}_{D(\Phi)} = \{A,B\}^{*} + i\{\{A,B\}^{*},(\mathcal{P}_j\xi_j)\}^{*}(\mathcal{P}_j\xi_j)\frac{(b\kappa + a)}{\bar{\beta}(\omega + m)}. \tag{22}$$

If now we take for $A', B'$ the variables $q, \pi, \psi$ from (15), then on account of (8) we find from (22)

$$\begin{aligned} &\{\psi_i,\psi_j\}_{D(\Phi)} = \{\xi_i,\xi_j\}^{*} = -i\delta_{ij}, \quad \{q_i,\pi_j\}_{D(\Phi)} = \{x_i,\mathcal{P}_j\}^{*} = -i\delta_{ij}, \\ &\{q_i,\psi_j\}_{D(\Phi)} = \{x_i,\xi_j\}^{*} = 0, \quad \{q_i,q_j\}_{D(\Phi)} = \{x_i,x_j\}^{*} = 0, \\ &\{\pi_i,\pi_j\}_{D(\Phi)} = gF_{ij}(x) - ig\partial_k F_{ij}\xi^k(\mathcal{P}_m\xi_m)\frac{(b\kappa + a)}{\bar{\beta}(\omega + m)} = gF_{ij}(q). \end{aligned} \tag{23}$$

These relations together with (12) prove that the variables $\tilde{x}_i, \tilde{p}_j, \tilde{\xi}_k$ are Newton-Wigner variables. The relations (23) for $D = 4$ coincide with similar relations obtained in [3]. Expressions of initial variables in terms of canonical ones are given by the following

$$\begin{aligned} x_i &= q_i - i\psi_i(\pi_j\psi_j)\frac{(a\kappa + b)}{\tilde{\alpha}(\Omega + m)}, \\ p_i &= \pi_i + igF_{ij}\psi_j(\pi_k\psi_k)\frac{(a\kappa + b)}{\tilde{\alpha}(\Omega + m)}, \\ \xi_i &= \psi_i + \pi_i(\pi_j\psi_j)\frac{(a\kappa + b)}{\tilde{\alpha}(\Omega + m)}, \end{aligned} \tag{24}$$

where $\tilde{\alpha} = -a\kappa\Omega + bm, \Omega = \sqrt{\pi_i^2 + m^2 + igF_{ij}\psi_i\psi_j}$.

Thus the variables $q_i, \pi_j, \psi_k$ have canonical Dirac brackets and hence it is convenient to quantize the theory in terms of these variables as it was done for $D = 4$ [3]. Using (24) and the relations

$$\xi_{D+1} = -\frac{a(\pi_j\psi_j)}{\tilde{\alpha}}, \quad \xi_O = \frac{b(\pi_j\psi_j)}{\tilde{\alpha}}, \tag{25}$$

one can deduce the expression for the physical hamiltonian of the spinning particle in the external electromagnetic field in $D = 2n$ in terms of canonical variables

$$H_{\text{phys}} = \Omega - g\kappa A_O - ig\kappa\frac{F_{Ok}\psi_k(\pi_j\psi_j)}{\Omega(\Omega + m)}. \tag{26}$$

It is worth mentioning that to quantize the theory by the Berezin, Marinov prescription [4] one must expand $\Omega$, which enters e.g. (26 ), in powers of $F_{ij}\psi_i\psi_j$, the expansion terminating in the order $\frac{D-2}{2}$. If however we are interested in the terms in the hamiltonian which after quantization will be of the order of $\hbar$, then taking into account that after quantisation $\psi_i \Rightarrow \sqrt{\frac{\hbar}{2}}\sigma_i$, we find for the quantum Hamiltonian the expression

$$\hat{H}_{phys} \rightleftarrows \tilde{\Omega} - g\kappa A_O - ig\kappa\hbar\frac{F_{Ok}\pi_j(\sigma_k\sigma_j - \sigma_j\sigma_k)}{4\tilde{\Omega}(\tilde{\Omega} + m)} + ig\hbar\frac{F_{ij}\sigma_i\sigma_j}{4\tilde{\Omega}}, \tag{27}$$

where $\tilde{\Omega} = \sqrt{\pi_i^2 + m^2}$, $\rightleftarrows$ denotes the Weyl correspondence between operators and their symbols. Since the expressions for the initial variables in terms of canonical variables are form invariant in all dimensions, the equations of motion for $D = 2n$ are similar to those for $D = 4$ found in [2][3].

Thus the pseudoclassical Foldy-Wouthuysen transformation significantly simplified the search for the canonical variables and enabled the quantisation of $D = 2n$ dimensional relativistic spinning particle in the external electromagnetic field .

### *Acknowledgment*

The authors wish to thank I.V.Tyutin for useful discussions. This research was partly supported by the grant YPI-1993 of the "Bundesminister für Forschung und Technologie", Federal Republic of Germany.

ГРИГОРЯН Г.В.,ГРИГОРЯН Р.П.
ПСЕВДОКЛАССИЧЕСКОЕ ПРЕОБРАЗОВАНИЕ ФОЛДИ-ВОТХОЙЗЕНА И
КАНОНИЧЕСКОЕ КВАНТОВАНИЕ D=2n-МЕРНОЙ РЕЛЯТИВИСТСКОЙ
СПИНОВОЙ ЧАСТИЦЫ ВО ВНЕШНЕМ ЭЛЕКТРОМАГНИТНОМ ПОЛЕ.

(на английском языке, перевод авторов)

Редактор Т.А.Балаян
Технический редактор А.С.Абрамян

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Подписано в печать 30.12.93 Формат 60×84ё16
Офсетная печать.Уч.изд.л. 0,5 Тираж 100 экз
Зак.тип.N 127 Индекс 3649

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Отпечатано в Ереванском физическом институте
Ереван 36,ул.Братьев Алиханян 2

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Препринт ЕФИ-1398(9)-93

ГРИГОРЯН Г.Е., ГРИГОРЯН Р.П.

# ПСЕВДОКЛАССИЧЕСКОЕ ПРЕОБРАЗОВАНИЕ ФОЛДИ-ВОТХОЙЗЕНА И КАНОНИЧЕСКОЕ КВАНТОВАНИЕ $D=2n$-МЕРНОЙ РЕЛЯТИВИСТСКОЙ СПИНОВОЙ ЧАСТИЦЫ ВО ВНЕШНЕМ ЭЛЕКТРОМАГНИТНОМ ПОЛЕ.

Проведено каноническое квантование $D=2n$-мерной дираковской спиновой частицы в произвольном внешнем электромагнитном поле.в калибровке, позволяющей описывать одновременно частицу и античастицу (как массивную, так и безмассовую) еще на классическом уровне. Квантование проведено в терминах канонических координат и импульсов типа Ньютона-Вигнера, для получения которых использовано псевдоклассическое преобразование Фолди-Вотхойзена. Обсуждается связь данной схемы квантования с картиной Блоунта для дираковской частицы во внешнем электромагнитном поле.

Ереванский физический институт.

Ереван 1993

ИНДЕКС 3649

Նախնատիպ ԵՖԻ-1398(9)-93

ԳՐԻԳՈՐՅԱՆ Գ.Կ., ԳՐԻԳՈՐՅԱՆ Ռ.Պ.

# ՖՈԼԴԻ-ՎՈՏՀՈՅՍԵՆԻ ԿԵՂԾ ԴԱՍԱԿԱՆ ՁԵՎԱՓՈԽՈՒԹՅՈՒՆԸ ԵՎ D=2n ՉԱՓԱԿԱՆ ՍՊԻՆՕՐՏՎԱԾ ՄԱՍՆԻԿԻ ԿԱՆՈՆԱՎՈՐ ՔՎԱՆՏԱՑՈՒՄԸ ԱՐՏԱՔԻՆ ԷԼԵԿՏՐԱՄԱԳՆԻՍԱԿԱՆ ԴԱՇՏՈՒՄ.

Արտաքին էլեկտրամագնիսական դաշտում գտնվող D=2n չափական Դիրակի սպինօրտված մասնիկի քվանտացումը կատարված է այնպիսի տրամաչափության ընտրությամբ, որը թույլ է տալիս նկարագրել միաժամանակ մասնիկը և հակամասնիկը (զանգվածով և անզանգված) դասական տեսության շրջանակներում: Կանոնավոր քվանտացումը իրականացված է Նյուտոն-Վիգների տիպի կոորդինատների և իմպուլսների միջոցով, որոնց ստանալու համար օգտագործվում է Ֆոլդի-Վոտհոյսենի կեղծ դասական ձևափոխությունը: Քննարկվում է քվանտացման տվյալ եղանակի կապը Բլոունտի պատկերի հետ արտաքին էլեկտրամագնիսական դաշտում գտնվող Դիրակի սպինօրտված մասնիկի համար:

Երևանի Ֆիզիկայի Ինստիտուտ

Երևան - 1993