Preprint YERPHI-1341(36)-91

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE

G.V. GRIGORYAN, R.P. GRIGORYAN

CANONICAL QUANTIZATION OF D=2n DIMENSIONAL RELATIVISTIC SPINNING PARTICLE

ЦНИИатоминформ
ЕРЕВАН - 1991

L

29- 16

We regret that some of the pages in this report may not be up to the proper legibility standards, even though the best possible copy was used for scanning

Նախնատիպ ԵրՖԻ-1341(36)-91

Գ.Վ.ԳՐԻԳՈՐՅԱՆ, Ռ.Պ.ԳՐԻԳՈՐՅԱՆ

ՌԵԼՅԱՏԻՎԻՍՏԱԿԱՆ D-ՉԱՓԱՅԻՆ ՍՊԻՆԱՅԻՆ ՄԱՍՆԻԿԻ ԿԱՆՈՆԱՎՈՐ ՔՎԱՆՏԱՑՈՒՄԸ

Դիրակի մասնիկի կանոնավոր քվանտացումը անց է կացվել այնպիսի չափարկումով, որը թույլ է տալիս նկարագրել կանգված և անկանգված մասնիկներ։ Քննարկումը տարվել է D=2n տարածաչափության համար։ Տեսության դինամիկ փոփոխականների համար դուրս են բերված դասական բանաձևեր, որոնց քվանտային նմանակները ավելի վաղ ստացել է Փրայսը։

Երևանի ֆիզիկայի ինստիտուտ

Երևան 1991

In papers [1] devoted to the problem of quantization of a point particle, it was shown, that the description of the electron spin is possible on the classical level if one introduces to theory a new type of variables - elements of Grassmann algebra. For quantization of this theories Dirac's method [2] of quantization of constrained systems was used, which prescribes to substitute the Dirac brackets of the variables of classical theory by the commutators (or anticommutators for odd elements of Grassmann algebra) of corresponding quantum operators. For odd elements Clifford algebra emerges which is realized through Dirac's γ-matrices. The first class constraints of the classical theory are transformed to equations on vectors, describing physical states- Dirac covariant equation in [1]. However this quantization scheme wasn't vigorously proved so far.

In this paper a different quantization scheme is applied. Namely, gauge fixing additional constraints, equal in number to a number of first-class constraints,is introduced into a classical theory.As a result only second-class constraints are present in the theory,and Dirac's brackets of this theory,consistent with all second class constraints, are computed for independent variables (for systematic treatment of the method see [3]). In a recent paper[4] this scheme was applied to the quantization of a relativistic particle with one of a gauge fixing constraints taken in a form $x_0 + \tau \, sign \, p_0 = 0$, which allows to describe particle and antiparticle at the same time in the classical theory. In that paper the quantization of relativistic spinning massive particle was also considered. However the obtained result does not allow to investigate the limit $m \to 0$, describing neutrino-like particles; this is a consequence of a definite form of the additional "fermion" gauge-fixing constraint in [4] .

In this paper the canonical quantization of the Dirac particle is carried out in a gauge, which allows to investigate both massive and massless particles. The investigation is carried out in spacetimes of $D=2n$ dimensions (while in [4] D=4). In sect 2. the complete set of constraints of the theory is obtained by Dirac's procedure. In sect.3 Dirac brackets for initial physical variables are calculated and new variables, for which the quantum commutation relations have a canonical form, are introduced. In sect. 4 the quantization of the theory is carried out; the classical Pauli-Lubanski vector is introduced and (within the Dirac's quantization scheme) the corresponding quantum operator is constructed; it's shown, that up to a constant, this opérator coincides with the operator, corresponding to the initial variable (more correctly, it's gauge invariant generalization) describing the spin degrees of freedom. In sect. 5 the limit $m \to 0$ of the theory is analysed. In sect. 6 the theory in this quantization scheme is shown to coincide with the Dirac theory of a spinning particle in Foldy-Wouthuysen representation.

In sect.7 several classical relations among the dynamical variables of the theory, quantum analogs of which were obtained in preceding sections, are deduced.

2. Consider the action of the theory, describing the relativistic massive spinning particle in a spacetime at any dimensions D=2n

$$S=\frac{1}{2}\int d\tau\left[\frac{(\dot{x}^{\mu})^2}{e}+em^2-i\left(\xi_{\mu}\dot{\xi}^{\mu}-\xi_{D+1}\dot{\xi}_{D+1}\right)-i\chi\left(\frac{\xi_{\mu}\dot{x}^{\mu}}{e}-m\xi_{D+1}\right)\right] \qquad (1)$$

Here $\mu=0,1,\dots,D-1$, $g_{\mu\nu}=(1,-1,\dots-1)$, x^{μ} – the particle coordinates, ξ^{μ}-grassmann variables, describing the spin degrees of freedom, ξ_{D+1}, χ and e are additional fields, e being an even, ξ_{D+1} and χ — odd elements at Grassmann algebra; the dot means the differentiation by τ parameter along the trajectory of the particle. Note, that the introduction of the additional field permits to consider a limit $m\to 0$ of the theory; the field χ is introduced as a Lagrange multiplier to the additional constraints, excluding from the theory indefinite metric states, while the field ξ_{D+1} - the analog of ξ_5 in four dimensions - is introduced to make the presence of the mass in that constraint possible. The action (1) is invariant under reparametrization and supergauge transformations [1,5]

$$\delta x^{\mu}=i\varepsilon\xi^{\mu}, \quad \delta e=i\varepsilon\chi, \quad \delta\chi=2\dot{\varepsilon}, \qquad (2)$$
$$\delta\xi^{\mu}=\varepsilon\left(\frac{\dot{x}^{\mu}}{e}-\frac{i}{2}\frac{\chi}{e}\xi^{\mu}\right)\equiv\varepsilon p^{\mu}, \quad \delta\xi_{D+1}=\varepsilon m.$$

From (1) we find canonical momenta, conjugated to coordinates x^{μ}, e, ξ^{μ}, ξ_{D+1}, χ :

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\dot{x}_\mu}{e} - \frac{i}{2}\chi\,\frac{\xi_\mu}{e},$$

$$\pi_e = \frac{\partial L}{\partial \dot{e}} = 0, \quad \pi_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu} = \frac{i}{2}\xi_\mu, \tag{3}$$

$$\pi_{D+1} = \frac{\partial L}{\partial \dot{\xi}_{D+1}} = -\frac{i}{2}\xi_{D+1}, \quad \pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0.$$

Relations (3), except the first one, are primary constraints

$$\Phi_\mu \equiv \pi_\mu - \frac{i}{2}\xi_\mu \approx 0, \quad \mu = 0, 1, \ldots, D-1 \tag{4}$$

$$\Phi_D \equiv \pi_{D+1} + \frac{i}{2}\xi_{D+1} \approx 0, \quad \Phi_{D+5} \equiv \pi_e \approx 0, \quad \Phi_{D+7} \equiv \pi_\chi \approx 0.$$

Here and below the indexing of constraints is chosen with the aim to make the matrix of Poisson brackets of all constraints $C_{ee'} = \{\Phi_e, \Phi_{e'}\}$ as compact as possible. The canonical Hamiltonian of the theory is

$$H = \dot{x}^\mu p_\mu + \dot{\xi}^\mu \pi_\mu + \dot{\xi}_{D+1}\pi_{D+1} - L = \tag{5}$$

$$= \frac{e}{2}(p^2 - m^2) + \frac{i}{2}\chi(p_\mu \xi^\mu - m\,\xi_{D+1}).$$

...g Dirac's procedure [2], we introduce total Hamiltonian ...he system:

$$H^* = H + i\lambda^\mu \Phi_\mu + i\lambda_D \Phi_D + \lambda_{D+5}\Phi_{D+5} + i\lambda_{D+7}\Phi_{D+7} \tag{6}$$

...are Lagrange multipliers) and find the secondary constraints

$$\phi_{D+1} \equiv p_\mu \xi^\mu - m\xi_{D+1} \approx 0, \quad \phi_{D+3} \equiv |p_0| - \omega \approx 0, \quad \omega = (\vec{p}^2 + m^2)^{\frac{1}{2}}. \tag{7}$$

The constraints $\phi_{D+3}, \phi_{D+5}, \phi_{D+7}$ are first class. There is one more first class constraint φ, which is a linear combination of the constraints $\phi_\mu, \phi_D, \phi_{D+1}$:

$$\varphi = p^\mu \phi_\mu + m\phi_D + i\phi_{D+1} =$$

$$= \frac{i}{2}\left(p^\mu \xi_\mu - m\xi_{D+1}\right) + p\pi + m\pi_{D+1} . \tag{8}$$

Hence there are four first class constraints in the theory and, according to general prescriptions [2,3], to eliminate the degeneration, one has to add four gauge fixing constraints. First we introduce two constraints (according to the number of primary first class constraints and will try to obtain remaining two other constraints from the requirement that the τ derivatives of these constraints are equal to zero: this method will ensure the consistency of new constraints with former equations of motions [2,3])

$$\phi_{D+4} \equiv x_0 + \tau \operatorname{sign} p_0 \approx 0, \quad \phi_{D+8} \equiv \chi \approx 0 \tag{9}$$

(according to constraint $\phi_{D+4} \approx 0$ $x_0 = \tau$ for sign $p_0 < 0$ and $x_0 = -\tau$ for sign $p_0 > 0$; the first case corresponds to a particles, the second - to antiparticles; for details see [4]). Next it's convenient to go over to canonical variables x'^μ, p'_μ connected with variables x^μ and p_μ by relations

$$x'^0 = x^0 + \tau \operatorname{sign} p_0, \quad x'^i = x^i, \quad p'_\mu = p_\mu . \tag{10}$$

In terms of new variables the constraint ϕ_{D+4} now becomes $X'^0 \approx 0$, while the total Hamiltonian, with new constraints (9) taken into account, is changed to

$$H_G^* = |P_0| + H + i\lambda^\mu \phi_\mu + i\lambda_D \phi_D + \lambda_{D+4}\phi_{D+4} + \tag{11}$$
$$+ \lambda_{D+5}\phi_{D+5} + i\lambda_{D+7}\phi_{D+7} + i\lambda_{D+8}\phi_{D+8} .$$

Conservation in time of the constraints (9) means that the following relations hold:

$$\dot{\phi}_{D+4} = \{X_0', H_G^*\} = eP_0 + \operatorname{sign} P_0 = 0 . \tag{12}$$
$$\dot{\phi}_{D+8} = \{X, H_G^*\} = -i\lambda_{D+7} = 0 .$$

The first of these relations is a new secondary constraint

$$\phi_{D+6} \equiv e + \frac{1}{|P_0|} \approx 0 . \tag{13}$$

This is the third of four gauge fixing constraints. It's easy to check, that the constraint ϕ_{D+6} is conserved in time and no new constraints follow.

At last we introduce the fourth additional constraint

$$\phi_{D+2} \equiv a\xi_0 + b\xi_{D+1} \approx 0 , \tag{14}$$

where a and b are parameters, which do not become zero simultaneously (note, that the theory depends only on the ratio a/b; $a=0$

corresponds to $\xi_{D+1} \approx 0$, while $a \to \infty$ corresponds to $\xi_0 \approx 0$; below an additional restriction on that ratio will be imposed). In paper [4], where the space time dimension was $D=4$, instead of (14) the constraint $\xi_5 \approx 0$ was chosen. For $m \neq 0$ the constraint (8) then becomes a second class. However for $m=0$ the constraint (8) remains a first class constraint and the degeneracy of the theory isn't eliminated. The additional relation (14) (with $a \neq 0$) is more suitable, since for both $m \neq 0$ and $m=0$ the theory isn't degenerated. Thus after the introduction of additional constraints (9), (13) and (14) all constraints in the theory are second class and we can now calculate Dirac brackets for independent dynamical variables. The physical Hamiltonian is given by $H_{ph} = |P_0| = (\vec{p}^2 + m^2)^{\frac{1}{2}}$. Taking into account the fact that the Dirac brackets have iterative property and that preliminary Dirac brackets with a second class constraints $\mathcal{X} \approx 0$, $\mathcal{T}_{\mathcal{X}} \approx 0$ are equal to Poisson bracket with remaining variables, we can omit these two constraints. Thus the complete set of constraints is the following

$$
\begin{aligned}
&\phi_\mu = \pi_\mu - \frac{i}{2}\xi_\mu, \quad \mu = 0, 1, \ldots, D-1, && \phi_{D+3} = |P_0| - \omega, \\
&\phi_D = \pi_{D+1} + \frac{i}{2}\xi_{D+1}, && \phi_{D+4} = X'_0, \qquad (15)\\
&\phi_{D+1} = p\xi - m\xi_{D+1}, && \phi_{D+5} = \pi_e, \\
&\phi_{D+2} = a\xi_0 + b\xi_{D+1}, && \phi_{D+6} = e + 1/|P_0|.
\end{aligned}
$$

3. Dirac brackets of two functions F , and G of canonical variables are defined as follows[2] :

$$\{F, G\}_D = \{F, G\} - \{F, \Phi_e\} C^{-1}_{ee'} \{\Phi_{e'}, G\}, \tag{16}$$

where $C^{-1}_{ee'}$ is a matrix inverse to $C_{ee'} = \{\Phi_e, \Phi_{e'}\}$, which is composed at Poisson brackets of second class constraints Φ_e. Here we present the results of the calculation of Dirac brackets (with constraints given by (15)) only for independent dynamical variables. In this theory we have chosen as independent $3D-3$ variables X^i, P_i, ξ^i (Dirac brackets for other variables of the theory are given in the appendix)

$$\{x^i, x^j\}_D = \frac{i\gamma}{\alpha^2}\left[\xi^i\xi^j + \frac{\xi_0}{P_0}\left(\xi^i p^j - \xi^j p^i\right)\right], \quad \{x^i, P_j\}_D = \delta^i_j,$$

$$\{x^i, \xi^j\}_D = \frac{\gamma}{\alpha^2}\xi^i p^j - \frac{\gamma}{\alpha^2}\frac{\xi_0}{P_0} p^i p^j, \quad \{\xi^i, \xi^j\}_D = -i\left(\delta^{ij} - \frac{p^i p^j}{\alpha^2}\gamma\right), \tag{17}$$

$$\{P_i, P_j\}_D = 0, \quad \{P_i, \xi^j\}_D = 0,$$

$$\alpha = aP_0 + bm, \quad \beta = am + bP_0, \quad \gamma = a^2 - b^2, \quad \xi^0 = \frac{b}{\beta}(\vec{P}\vec{\xi}).$$

The relations (17) allow in principle to turn to quantization of the theory, as it was done in [4]. However it seems more appropriate to find new variables q^i and ψ^i, for which Dirac brackets will have a more natural form:

$$\{q^i, q^j\}_D = 0, \quad \{\psi^i, \psi^j\}_D = -i\delta^{ij}. \tag{18}$$

We will suppose, that q^i and ψ^i are given by the expressions

$$q^i = x^i + f(\vec{p})\xi^i(\vec{p}\vec{\xi}), \quad \psi^i = \xi^i + \varphi(\vec{p})p^i(\vec{p}\vec{\xi}), \tag{19}$$

where $f(\vec{p})$ an $\varphi(\vec{p})$ are some functions, depending only on spatial components of the momentum. Substituting (19) into (18) and using (17) we find that $f=-i(a+b\varkappa)/\beta(\omega+m)$, $\varphi=(a+b\varkappa)/\beta(\omega+m)$, where $\varkappa=-\operatorname{sign} p_0$. Thus

$$q^i=\chi^i-i\frac{a+b\varkappa}{\beta(\omega+m)}\xi^i(\vec{p}\vec{\xi}),\quad \psi^i=\xi^i+\frac{a+b\varkappa}{\beta(\omega+m)}p^i(\vec{p}\vec{\xi}). \quad (20)$$

The expressions, inverse to (20), have the form

$$\chi^i=q^i-i\frac{a\varkappa+b}{\alpha(\omega+m)}\psi^i(\vec{p}\vec{\psi}),\quad \xi^i=\psi^i+\frac{a\varkappa+b}{\alpha(\omega+m)}p^i(\vec{p}\vec{\psi}). \quad (21)$$

It's easy to show, that if parameters a and b are subjected to a condition $a/b\varkappa\in(-\infty,0]$, then (20) and (21) do not contain singularities i.e. α and β are different from zero for all values of $\vec{p}$. Note that the singularity in (21) for $\alpha=0$ is connected with the fact, that for $\alpha=0$ the constraint Φ_{D+2} becomes a supergauge invariant quantity: $\Phi_{D+2}\big|_{\alpha=0}=a\tilde{\xi}_0=$ $=a[\xi_0-(p_0/m)\xi_{D+1}]$, and thus does not eliminate the degeneracy of the theory. From (20) and (21) it is easy to see, that on the gauge $\xi_0-\varkappa\xi_{D+1}$ $(a=1, b=-\varkappa)$ χ^i and ξ^i are canonical variables. In other words, in that gauge Dirac brackets coincide with Poisson brackets; this is due to the possibility of performing canonical transformation of constraints

$$\pi_0-\frac{i}{2}\xi_0\approx 0,\quad \pi_{D+1}+\frac{i}{2}\xi_{D+1}\approx 0, \quad (22)$$

$$p\xi-m\xi_{D+1}\approx 0,\quad \xi_0-\varkappa\xi_{D+1}\approx 0$$

to a special form [3], when half of them in terms of new variables have the form - canonical variable is equal to zero. Note also that in the gauge $\xi_5\approx 0$ $(a=0)$ the expressions (20), (21) coincide with similar expressions in [4].

It's not difficult to calculate remaining Dirac brackets of variables q^i, ψ^i:

$$\{q^i, p_j\}_D = \delta^i_j, \quad \{q^i, \psi^j\}_D = 0, \quad \{p_i, \psi^j\}_D = 0. \tag{23}$$

It's evident, that it's convinient to start the quantization of the theory by quantization of expressions (18),(23).

4. The transition to a quantum theory is preformed by the replacement of the canonical coordinates and momenta by operators, for which the commutators (anticommutators) are defined by the rule $[\,,\,] = i\hbar\{\,,\,\}_D$. Then we get from (18) and (23) the commutation relations.

$$[\hat{q}^i, \hat{q}^j]_- = [\hat{p}_i, \hat{p}_j]_- = [\hat{q}^i, \hat{\psi}^j]_- = [\hat{p}_i, \hat{\psi}^j]_- = 0, \tag{24}$$

$$[\hat{q}^i, \hat{p}_j]_- = i\hbar\delta^i_j, \quad [\hat{\psi}^i, \hat{\psi}^j]_+ = \hbar\delta^{ij}. \tag{25}$$

Consider the second of the relations (25). It generates Clifford algebra in (D-1) space dimensions. As is well known, only one finite dimensional irreducible representation at this algebra by the matrices (denoted σ^i) with a dimension $2^{(D-2)/2} \times 2^{(D-2)/2}$:

$$\hat{\psi}^i = \pm(\hbar/2)^{\frac{1}{2}}\sigma^i = \varkappa(\hbar/2)^{\frac{1}{2}}\sigma^i, \quad i = 1, \dots, D-1. \tag{26}$$

Using the relation (21), together with the equality $\xi_{D+1} = -\frac{a}{\beta}(\vec{p}\vec{\xi}) = (\varkappa a/\alpha)(\vec{p}\vec{\psi})$, which follows from constraints Φ_{D+1}, Φ_{D+2} and (21), we can write down the expressions for operators, corresponding to X^i, ξ^i, ξ_{D+1}:

$$\hat{X}^i = \hat{q}^i - \frac{i\hbar}{4}\frac{a\varkappa + b}{\hat{\alpha}(\hat{\omega}+m)}[\sigma^i, (\hat{\vec{p}}\vec{\sigma})], \quad \hat{\xi}_{D+1} = a\left(\frac{\hbar}{2}\right)^{\frac{1}{2}}\frac{\hat{\vec{p}}\vec{\sigma}}{\hat{\alpha}},$$

$$\hat{\xi}^i = \varkappa\left(\frac{\hbar}{2}\right)^{\frac{1}{2}}\left[\sigma^i + \frac{a\varkappa + b}{\hat{\alpha}(\hat{\omega}+m)}\hat{p}^i(\hat{\vec{p}}\vec{\sigma})\right], \quad \hat{\alpha} = a\hat{p}_0 + bm. \tag{27}$$

Here $\hat{q}^i$ is the operator of "physical" coordinate. The physical operator for spin variables $\hat{\tilde{\xi}}^\mu$ corresponds to a classical variable $\tilde{\xi}^\mu = \xi^\mu - (p^\mu/m)\xi_{D+1}$. The latter is gauge invariant and is D-dimensional analogy of the spin variable $\tilde{\xi}^\mu (\mu=0,\dots 3)$, in terms of which the spin tensor is [5]

$$S^{\mu\nu} = i\tilde{\xi}^\mu\tilde{\xi}^\nu = i\xi^\mu\xi^\nu + \frac{i}{m}\xi_5(\xi^\mu p^\nu - \xi^\nu p^\mu), \quad \tilde{\xi}^\mu = \xi^\mu - \frac{p^\mu}{m}\xi_5 \tag{28}$$

(In paper [6] it was noted, that the Dirac bracket algebra of x^i and S^i (vector dual to S^{ij} in D=4 dimensions), which is called spin vector- coincides with the algebra of mean position and spin operators, introduced by Pryce, however a "bad" gauge fixing $x_0=\tau$ was used and the operator realization of x^i and ξ^i wasn't discussed). The invariance of $\tilde{\xi}^\mu$ under transformations (2) ensures the fermion gauge independence of spin vector. Note that the canonical generator of Lorents rotations

$$\mathcal{J}^{\mu\nu} = -(x^\mu p^\nu - x^\nu p^\mu + i\xi^\mu\xi^\nu), \tag{29}$$

which for D=4 is identified with a total angular momentum of the particle, is also invariant under transformations (2) [5] For operator $\hat{\tilde{\xi}}^\mu$ we have

$$\hat{\tilde{\xi}}^0 = -\left(\frac{\hbar}{2}\right)^{\frac{1}{2}}\frac{\hat{\vec{p}}\vec{\sigma}}{m}, \quad \hat{\tilde{\xi}}^i = \varkappa\left(\frac{\hbar}{2}\right)^{\frac{1}{2}}\left[\sigma^i + \frac{\hat{\vec{p}}^i(\hat{\vec{p}}\vec{\sigma})}{m(\hat{\omega}+m)}\right]. \tag{30}$$

Using (27), it's easy to express $\hat{\mathcal{J}}^{\mu\nu}$ in terms of operators of physical variables:

$$\hat{\mathcal{J}}^{ik} = -\hat{q}^i\hat{p}^k + \hat{q}^k\hat{p}^i - \frac{i\hbar}{4}[\sigma^i, \sigma^k]_-$$

$$\hat{\mathcal{J}}^{0k} = -x^0\hat{p}^k - \frac{1}{2}\varkappa[\hat{q}^k, \hat{\omega}]_+ - \frac{i\hbar\varkappa}{4(\hat{\omega}+m)}\hat{p}^j[\sigma^k, \sigma^j]_- \tag{31}$$

Apart from operators considered above, it seems important to construct the Pauli- Lubanski vector within this quantization scheme. We define the Pauli-Lubanski vector in classical theory in D-dimensions by

$$W_\mu = \frac{1}{(D-2)!}\varepsilon_{\mu\nu\lambda_2\ldots\lambda_{D-1}} P^\nu J^{\lambda_2\lambda_3}\ldots J^{\lambda_{D-2}\lambda_{D-1}} \tag{32}$$

where $\varepsilon_{\mu\nu\lambda_2\ldots\lambda_{D-1}}$ is totally antisymmetric tensor in D-dimensions and $J^{\mu\nu}$ is given by (29). Substituting from (29) in (32) we obtain

$$W_\mu = \frac{(-i)^{\frac{D-2}{2}}}{(D-2)!}\varepsilon_{\mu\nu\lambda_2\ldots\lambda_{D-1}} P^\nu \xi^{\lambda_2}\ldots\xi^{\lambda_{D-1}} \tag{33}$$

It's easy to show, that for W_μ the relation $W_\mu(\xi^\nu) = W_\mu(\tilde{\xi}^\nu)$ holds, from which it follows that W_μ is fermion gauge invariant. Recalling (27), (33) implies for $\hat{W}_\mu$ operator

$$\begin{aligned}\hat{W}_0 &= \frac{(-i)^{\frac{D-2}{2}}}{(D-2)!}\varepsilon_{0ij_2\ldots j_{D-1}}\hat{P}^i\hat{\xi}^{j_2}\ldots\hat{\xi}^{j_{D-1}} = \\ &= \frac{(-i)^{\frac{D-2}{2}}}{2^{(D-2)/2}(D-2)!}\varepsilon_{ij_2\ldots j_{D-1}}\hat{P}_i\sigma_{j_2}\cdots\sigma_{j_{D-1}} = -\left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}}(\vec{\hat{P}}\vec{\sigma}), \\ \hat{W}_i &= \frac{(-i)^{\frac{D-2}{2}}}{(D-2)!}\varepsilon_{i0\lambda_2\ldots\lambda_{D-1}}\hat{P}^0\hat{\xi}^{\lambda_2}\ldots\xi^{\lambda_{D-1}} = \\ &= \varkappa\left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}}\left(m\sigma_i + \frac{\hat{P}_i(\vec{\hat{P}}\vec{\sigma})}{\hat{\omega}+m}\right)\end{aligned} \tag{34}$$

The operator $\hat{W}_\mu$ is naturally supergauge invariant. In deducing (34) we used the relation $\varepsilon_{01\ldots D-1} = -\varepsilon_{12\ldots D-1} = -1$ and also the σ_i -matrix algebra in $(D-1)$ dimensions [8]. Comparing (34) and (30) we find the relation between the operators $\hat{W}_\mu$ and $\hat{\tilde{\xi}}_\mu$:

$$\hat{W}_\mu = \left(\frac{\hbar}{2}\right)^{\frac{D-3}{2}} m\,\hat{\tilde{\xi}}_\mu . \tag{35}$$

This relation means, that one and the same quantum operator corresponds to two different classical objects $m\tilde{\xi}_\mu$ and W_μ. Perhaps, with a correct definition of classical average [1a] this correspondence is true in the classical theory as well. This question needs further investigations.

5. Let us now discuss the limit $m=0$ of the action (1). As we already mentioned in sect. 2 the fermion gauge fixing in the form $a\xi_0 + b\xi_{D+1} \approx 0$ (with $a \neq 0$), unlike the case of $\xi_{D+1} \approx 0$, allows to consider the massless theory. Hence the results for canonical quantization of massless theory coincide with results of preceding sections with $m=0$. From (34) we have

$$W_\mu \Big|_{m=0} = \mathscr{X} \left(\frac{\hbar}{2} \right)^{\frac{D-2}{2}} \hat{p}_\mu \frac{(\hat{\vec{p}}\,\vec{\sigma})}{|\hat{\vec{p}}|} \tag{36}$$

This expression, which, by the way, is correct in any gauge because of the gauge invariance of W_μ, for D = 4 coincides with a well-known expression for Pauli-Lubanski vector of massless theory [9]. The presence of the factor $\mathscr{X} = -\operatorname{sign} p_0$ reflects the fact that helicites of particle and antiparticle have opposite signs. From (35) it follows, that the physical meaning must be attributed to $m\hat{\tilde{\xi}}_\mu$ which in the limit $m \to 0$ is equal to

$$m\hat{\tilde{\xi}}_\mu \Big|_{m=0} = -\hat{p}_\mu \hat{\xi}_{D+1} \tag{37}$$

Comparing this expression with (36) one can see, that on quantum level $\hat{\xi}_{D+1}$ corresponds to helicity operator of a massless particle. Usually in theories of massless particle the summand $\xi_{D+1}\dot{\xi}_{D+1}$ is omitted from the action, because for $m=0$ no vertices including ξ_{D+1} exist. However our investigation shows, that this term must be included in the action and after quantization $\hat{\xi}_{D+1}$ must be interpreted as an massless particle helicity operator. Note, also, that as it follows from (2), the variable ξ_{D+1} for $m=0$ is supergauge invariant.

6. We'll demonstrate now, that operator $\hat{J}^{\mu\nu}$ and $\hat{W}_{\mu}$ constructed above, coincide with the corresponding operators of Dirac theory in Foldy – Wouthuysen (F W) representation.

The operator $\hat{A}^{FW}$ in FW representation is defined by relations

$$\hat{A}^{FW} = U\hat{A}^{D}U^{-1}, \quad U = [2\hat{\omega}(m+\hat{\omega})]^{-\frac{1}{2}}(m+\hat{\omega}+\vec{\gamma}\hat{\vec{P}}), \qquad (38)$$

where $\hat{A}^{D}$ is operator in Dirac representation. We choose for γ-matrices the representation

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma_{i} \\ -\sigma_{i} & 0 \end{pmatrix}, \quad \gamma_{D+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (39)$$

We have for operator $\hat{J}^{\mu\nu}$ in Dirac theory the expression

$$\hat{J}^{\mu\nu}_{D} = \hat{X}^{\mu}\hat{P}^{\nu}_{D} - \hat{X}^{\nu}\hat{P}^{\mu}_{D} + \frac{\hbar}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]. \qquad (40)$$

(here $\hat{X}^{\mu}$-physical coordinate operator, $P^{\mu}_{D} = -P^{\mu}$). It's easy to check, that the spatial part of total angular momentum $\hat{J}^{ik}_{D}$ is invariant under FW transformation: $\hat{J}^{ik}_{FW} = \hat{J}^{ik}_{D}$. Making use of the γ- matrices representation (39) we get from (40)

$$\hat{J}^{ik}_{FW} = -\hat{X}^{i}\hat{P}^{k} + \hat{X}^{k}\hat{P}^{i} - \frac{i\hbar}{4}[\sigma^{i}, \sigma^{k}]. \qquad (41)$$

As for the operator $\hat{J}^{0k}$, in the Dirac picture $(P_{0} = -\gamma^{0}\gamma^{i}p^{i} + \gamma^{0}m)$ it can be transformed to the form

$$\hat{J}^{0k}_{D} = -X^{0}\hat{P}^{k} - \frac{1}{2}[\hat{X}^{k}, \hat{P}^{0}]_{+} . \qquad (42)$$

Taking into account, that $\hat{P}^{0}_{FW} = \gamma^{0}\hat{\omega}$ it's easy to find $\hat{J}^{0k}$ in FW-representation

$$\hat{J}^{0k}_{FW} = -X^{0}\hat{P}^{k} - \frac{1}{2}\gamma^{0}\{\hat{X}^{k}, \hat{\omega}\} - \frac{i\hbar}{4}\frac{\hat{P}^{j}\gamma^{0}}{\hat{\omega}+m}[\sigma^{k}, \sigma^{j}] \qquad (43)$$

Comparing (41) and (43) with corresponding expressions in (31) we find out that they coincide; the upper component of (43) corresponds to $\mathscr{X} = 1$, while the lower components correspond to $\mathscr{X} = -1$. We define Pauli-Lubanski vector $\hat{W}_\mu^D$ in D-dimensional Dirac theory as a quantum analogy of (32) where $\hat{\mathcal{J}}_D^{\mu\nu}$ is given by (40). Using the explicit expression for $\hat{\mathcal{J}}_D^{\mu\nu}$ and using the relation

$$\frac{i^{D/2}}{(D-2)!}\varepsilon_{\mu\nu\lambda_2\ldots\lambda_{D-1}}\gamma^{\lambda_2}\ldots\gamma^{\lambda_{D-1}} = \sigma_{\mu\nu}\gamma_{D+1}, \quad \gamma_{D+1} = i^{\frac{D-2}{2}}\gamma^0\ldots\gamma^{D-1}, \tag{44}$$

we find that $\hat{W}_\mu^D$ is given by

$$\hat{W}_\mu^D = i\left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}}\gamma_{D+1}\sigma_{\mu\nu}\hat{P}^\nu \tag{45}$$

For $\hat{W}_\mu^D$ we have $\hat{P}^\mu \hat{W}_\mu = 0$, while it's square is the Casimir invariant of Poincare group in D-dimensions

$$\begin{gathered}\hat{W}_\mu^D \hat{W}_D^\mu = -\left(\frac{\hbar}{2}\right)^{D-2} m^2 \Sigma^i \Sigma^i = -\left(\frac{\hbar}{2}\right)^{D-2} m^2 (D-1) \\ \Sigma^i = \gamma_{D+1}\gamma^0\gamma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \Sigma^i\Sigma^i = D-1 .\end{gathered} \tag{46}$$

Calculating $\hat{W}_\mu$ in FW-representation we come to the expressions

$$\begin{gathered}\hat{W}_0^{FW} = -\left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}} \Sigma^i \hat{P}_i , \\ \hat{W}_i^{FW} = -\left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}} m\gamma^0\left(\Sigma^i + \hat{P}_i \frac{\hat{P}_j \Sigma^j}{m(\hat{\omega}+m)}\right)\end{gathered} \tag{47}$$

Taking into account the explicit representation of γ^0 and the definition of Σ^i through σ-matrices, we convince ourselves that (47) and (34) coincide. Again the upper components of (47) correspond to the value $\mathscr{X} = 1$, and the lower ones — $\mathscr{X} = -1$.

Thus upon the canonical quantization operator $\hat{\xi}_\mu$, corresponding to a grassmann variable $\tilde{\xi}_\mu$, is proportional to Pauli-Lubanski pseudovector in FW representation, while $\hat{H}^{FW} = \hat{H}_{ph}$ [10] (H^{FW} - Hamiltonian of Dirac theory in FW representation)

7. In this section we will discuss the generalization of several pseudoclassical fourdimensional quantities to D-dimensional spacetime.

Formula (28) gives the expression of a tensor $S^{\mu\nu}$, which is called a classical spin tensor, while (32) and (29) define the classical Pauli-Lubanski pseudovector. Consider now D-dimensional classical pseudotensor

$$S^{\tilde{\xi}*}_{\mu\nu} = \frac{(-i)^{(D-2)/2}}{(D-2)!}\varepsilon_{\mu\nu\lambda_2\lambda_3\ldots\lambda_{D-1}}\tilde{\xi}^{\lambda_2}\ldots\tilde{\xi}^{\lambda_{D-1}}. \tag{48}$$

Using the latter we will deduce on the classical level several relations, operator analogies for which are known in quantum theory. The tensor $S^{\tilde{\xi}*}_{\mu\nu}$ for $D=4$ is dual to spin tensor (28), and $S^{\tilde{\xi}*}_{oi}$ is a spin vector in terms of $\tilde{\xi}$: $S^{\tilde{\xi}*}_{oi} = -\frac{1}{2}\varepsilon_{ijk}\tilde{\xi}_j\tilde{\xi}_k = S^{\tilde{\xi}}_i$. Note also, that $S^{\tilde{\xi}*}_{\mu\nu}$ is supergauge invariant by construction. Consider now the components

$$S^{\tilde{\xi}*}_{oi} = -\frac{i^{(D-2)/2}}{(D-2)!}\varepsilon_{ij_2\ldots j_{D-1}}\tilde{\xi}^{j_2}\ldots\tilde{\xi}^{j_{D-1}}. \tag{49}$$

Using the relation $\tilde{\xi}^j = \psi^j + p^j(\vec{p}\vec{\psi})/m(\omega+m)$, which follows from the definition of $\tilde{\xi}$ and (21), we find from (49), that

$$S^{\tilde{\xi}*}_{oi} = \frac{\omega}{m}S^{\psi*}_{oi} - \frac{p_i}{m(\omega+m)}\left(p_j S^{\psi*}_{oj}\right), \tag{50}$$

where

$$S^{\psi*}_{oi} = -\frac{i^{(D-2)/2}}{(D-2)!}\varepsilon_{ij_2\ldots j_{D-1}}\psi^{j_2}\ldots\psi^{j_{D-1}}. \tag{51}$$

Introducing a notation $S^*_{oi} = S_i$ (in analogy with relatio[illegible] D =4) from (49) we get

$$S_i^{\tilde{\xi}} = \frac{\omega}{m} S_i^{\psi} - \frac{P_i (\vec{P}\vec{S}^{\psi})}{m(\omega+m)} . \qquad (52)$$

In terms of the pseudotensor $S_{\mu\nu}^{\tilde{\xi}*}$ the expression for the classical pseudotensor W_μ now has the form

$$W_\mu = S_{\mu\nu}^{\tilde{\xi}*} P^\nu . \qquad (53)$$

Using the equality $\tilde{\xi}_o = P_o(\vec{P}\vec{\psi})/\omega m$, on account of (52) we find

$$W_o = - P_i S_i^{\psi} ,$$

$$W_i = \operatorname{sign} P_o \cdot \left(m S_i^{\psi} + P_i (\vec{P}\vec{S}^{\psi}) / (\omega+m) \right) . \qquad (54)$$

The quantum analogy of formula (52) is obtained on account of (51) and the expression (26) for $\hat{\psi}^i$:

$$\hat{S}_i^{\tilde{\xi}} = \frac{\omega}{m} \hat{S}_i^{\psi} - \frac{\hat{P}_i (\hat{\vec{P}}\hat{\vec{S}}^{\psi})}{m(\omega+m)} , \quad \hat{S}_i^{\psi} = (-1)^{\frac{D}{2}} \left(\frac{\hbar}{2}\right)^{\frac{D-2}{2}} \sigma_i . \qquad (55)$$

For $D=4$ $\hat{S}_i^{\psi} = (\hbar/2)\sigma_i$ and (53) coincides with well known expression for spin operator, introduced in [7]. After quantization S_i^{ψ} is replaced by $\hat{S}_i^{\psi}$ (given by (55)), then the relation (54) turns into (34) .

Note, that here also, one and the same quantum operator corresponds to two different classical objects S_i^{ψ} and ψ_i .

Authors are indepted to Tyutin I.V. for useful discussions.

Appendix

Here we give Dirac brackets for initial variables of the theory, which were not given in the main text.

$$\{\xi_0,\xi_0\}_D=-i\,\frac{b^2(P_0^2-m^2)}{\alpha^2},\qquad \{\xi^i,\xi_0\}_D=-\frac{ib\beta P^i}{\alpha^2}$$

$$\{\xi_{D+1},\xi_0\}_D=-i\,\frac{ab}{\alpha^2}(m^2-P_0^2),\qquad \{\xi_{D+1},\xi^i\}_D=\frac{ia\beta}{\alpha^2}P^i,$$

$$\{\xi_{D+1},\xi_{D+1}\}_D=-i\,\frac{a^2}{\alpha^2}(P_0^2-m^2).$$

References

1. a). Berezin F.A., Marinov M.S., JETPh Letters 21 (1975) 320, Ann.Phys (N.Y.) 104 (1977) 336.

   b). Casalbuoni R., Nuovo Cim., 33A (1976) 115
   Barducci A. et al. Nuovo Cim. 35A (1976) 377
   Brink L. et al., Phys.Lett. 64B (1976) 435
   Brink L. et al. Nucl. Phys. B118 (1977) 435
2. Dirac P.A.M., Lectures on Quantum Mechanics, Belfer Graduate School of Science (Yeschiva University Press, New York, 1964).
3. Gitman D.M., Tyutin I.V., Quantization of Fields with Constraints, (Springer Verlag, Berlin, 1990).
4. Gitman D.M., Tyutin I.V., JETPh Letters 51 (1990) 188
5. Galvao C., Teitelboim C., J.Math. Phys. 21 (1980) 1863
6. Bachas C., Phys.Lett. B29 (1978) 62
7. M.H.L.Pryce, Proc.Roy.Soc. A195 (1948) 62
8. K.M.Case, Phys.Rev. 97 (1955) 810
9. Shirokov Y.M. JETPh 21 (1951) 748, DAN USSR 94 (1954) 857
10. L.L.Foldy, S.A.Wouthuysen Phys.Rev. 78 (1950) 29

The manuscript was received 20 December 1990

Г.В.ГРИГОРЯН,Р.П.ГРИГОРЯН

КАНОНИЧЕСКОЕ КВАНТОВАНИЕ D -МЕРНОЙ РЕЛЯТИВИСТСКОЙ СПИНОВОЙ ЧАСТИЦЫ

(на английском языке,перевод авторов)

Редактор Л.П.Лукаян
Технический редактор А.С.Абрамян

Подписано в печать I6/XII-9Iг. Формат 60x84/I6
Офсетная печать.Уч.изд.л. 0,8 Тираж 299 экз. Ц.IOк.
Зак.тип.№ I79 Индекс-3649

Отпечатано в Ереванском физическом институте
Ереван 36, ул.Братьев Алиханян,2

The address for requests:
Information Department
Yerevan Physics Institute
Alikhanian Brothers 2,
Yrevan, 375036
Armenia, USSR

ИНДЕКС 3649

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ