Preprint YERPHI-1342(38)-91

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
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SOME NEW SPINOR REPRESENTATIONS OF QUANTUM GROUPS Bq(n), Cq(n), Gq(2).

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ЦНИИатоминформ
ЕРЕВАН - 1991

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Նախնատիպ ԵՐՖԻ-1342(38)-91

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Bq(n), Cq(n), Gq(2) ՔՎԱՆՏԱՅԻՆ ԽՄԲԵՐԻ ՈՐՈՇ ՆՈՐ ՍՊԻՆՈՐԱՅԻՆ ՆԵՐԿԱՅԱՑՈՒՄՆԵՐԸ

Օգտագործելով Aq(n),Dq(n) քվանտային հանրահաշիվների հայտնի իրականացումը քվանտային Կլիֆֆորդի հանրահաշվով և 'folding" պրոցեդուրան մենք կառուցում ենք Bq(n), Cq(n), Gq(2) քվանտային հանրահաշիվների որոշ նոր սպինորային ներկայացումներ։

Երևանի ֆիզիկայի ինստիտուտ
Երևան 1991

1. INTRODUCTION

The progress of the quantum inverse scattering method [3] has been led Jimbo [8] and Drinfeld [2] to the definition of new algebraic structures known as quantum groups. They defined a Hopf algebras which can be considered as a quantum deformations of enveloping algebras.

The vertex operator representations of quantum affine algebras are constructed in articles [1] ,[5] . The spinor and oscillator representation of quantum enveloping algebras are presented in T. Hayashi's article [7] , where he defines the quantum analogs of Clifford and Weyl algebras with their algebra homomorphisms to the q-analogs of A(n),B(n), C(n) and D(n) algebras. In the article [4] the vertex operator representations of the twisted affine algebras were constructed using the folding procedure of Lee algebras [6] . Our purpose in this paper is to construct some new spinor representations of q-analogs of B(n),C(n),G(2) using the folding of Lee algebras [6] ,[4] . An oscillator representation of $SL_q(2)$ is also constructed.

In sec. 2 we recall the definition of the folding procedure in Lee algebras. In sec. 3 using the folding procedure we construct the spinor representation of $B_q(n)$ and $G_q(2)$ from $D_q(n+1)$ and $D_q(4)$ correspondingly. The spinor representations of $C_q(n)$ and $B_q(n)$ from the $A_q(2n-1)$ and $A_q(2n)$ are constructed in sec. 4 and 5.

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$$(6)\qquad \sum_{\upsilon=0}^{1-a_{ij}} (-1)^{\upsilon} \left[{1-a_{ij} \atop \upsilon} \right]_q e_i^{1-a_{ij}-\upsilon}\, e_j\, e_i^{\upsilon} = 0 \qquad (i \neq j)$$

$$\sum_{\upsilon=0}^{1-a_{ij}} (-1)^{\upsilon} \left[{1-a_{ij} \atop \upsilon} \right]_q f_i^{1-a_{ij}-\upsilon}\, f_j\, f_i^{\upsilon} = 0 \qquad (i \neq j)$$

(Here $\left[{m \atop n} \right]_q = \dfrac{[m]_q!}{[m-n]_q!\,[n]_q!}$, $\quad [m]_q! = [1]_q [2]_q \ldots [m]_q$)

From (5) follows:

$$(7)\qquad q^{h_i} e_j = e_j\, q^{h_i + a_{ij}} \qquad\qquad q^{h_i} f_j = f_j\, q^{h_i - a_{ij}}$$

Generators a_i, a_i^{+}, N_i construct a quantum Clifford algebra, if [7]:

$$(8)\qquad \begin{array}{lll} a_i^{+} a_i = [N_i]_q & a_i a_i^{+} = [1-N_i]_q & [N_i, a_j^{\mp}] = \mp \delta_{ij}\, a_j^{\mp} \\ [N_i, N_j] = 0 & a_i^{\mp} a_j^{\mp} = -\, a_j^{\mp} a_i^{\mp} & \end{array}$$

In [7] the quantum Clifford and Weyl algebra representation of quantum Lee algebras $A_q(n)$, $B_q(n)$, $C_q(n)$, $D_q(n)$, was constructed. For $D_q(n+1)$ it had the form:

$$(9)\qquad \begin{array}{lll} e_i = a_i\, a_{i+1}^{+} & f_i = a_{i+1}\, a_i^{+} & h_i = N_{i+1} - N_i \quad (i=1..n) \\ e_{n+1} = a_n\, a_{n+1} & f_{n+1} = a_{n+1}^{+}\, a_n^{+} & h_{n+1} = 1 - N_{n+1} - N_n \end{array}$$

Our aim is to try to extend the folding $D(n+1) \Rightarrow B(n)$ and $D(4) \Rightarrow G(2)$ to the quantum case and to use the representation (9) for $D_q(n+1)$ to construct new quantum Clifford algebra representations of $B_q(n)$ and $G_q(2)$.

1) $D(n+1) \Rightarrow B(n)$. We try to find a quantum analog of (3) in the form:

$$(10)\qquad \begin{array}{ll} e_n' = q^{1/2\, h_{n+1}}\, e_n + q^{-1/2\, h_n}\, e_{n+1} & e_i' = e_i \\ f_n' = f_n\, q^{1/2\, h_{n+1}} + f_{n+1}\, q^{-1/2\, h_n} & f_i' = f_i \end{array}$$

$$h_n' = h_n + h_{n+1} \qquad h_i' = h_i \qquad (i=1..n-1)$$

Proposition. Elements $e_1'..e_n'$, $f_1'..f_n'$, $h_1'..h_n'$ satisfy the properties:

(11) $[h_i', e_j'] = a_{ij}\, e_j' \qquad [h_i', f_j'] = -\, a_{ij}\, f_j'$

(12) $[e_i', f_i'] = [h_i']_q$ Here a_{ij} is Cartan matrix of B(n):

$$a_{ij} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & .. & & \\ & & .. & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

Formulas (11) are evident and the only nontrivial case in (12) is when i=n: $[e_n', f_n'] =$

$$= [\, q^{1/2 h_{n+1}} e_n + q^{-1/2 h_n} e_{n+1},\ f_n q^{1/2 h_{n+1}} + f_{n+1} q^{-1/2 h_n}] =$$

$$= q^{h_{n+1}} [e_n, f_n] + q^{-h_n} [e_{n+1}, f_{n+1}] +$$

$$+ q^{1/2(h_{n+1}-h_n-1)} [e_{n+1}, f_n] + q^{1/2(h_{n+1}-h_n+1)} [e_n, f_{n+1}] =$$

$$= q^{h_{n+1}} [h_n]_q + q^{-h_n} [h_{n+1}]_q = [h_n + h_{n+1}]_q = [h_n']_q$$

In the derivation we use (5),(7),(10). Unfortunately (10) do not generate Bq(n): the equations $[e_n', f_{n-1}'] = 0$ and $[e_{n-1}', f_n'] = 0$ are not satisfied. But if we take the quantum Clifford representation (9) of Dq(n+1) then they are satisfied:

$$[e_n', f_{n-1}'] = [q^{1/2 h_{n+1}} e_n, f_{n-1}] + [q^{-1/2 h_n} e_n, f_{n-1}] = 0$$

because $e_n \cdot f_{n-1} = a_n \cdot a_{n+1}^+ \cdot a_{n-1}^+ \cdot a_n = 0$ etc. In other words, in representation (9) all products $e_i \cdot f_j$ for neighboring i and j vanish.

Now consider the quantum Serre relations (6) among the generators e_i' (f_i'). All terms of type $e_i'^{\,1-a_{ij}-\upsilon}\ e_j'\ e_i'^{\,\upsilon}$ vanish ($i\neq j$). For example,

$$e_i'^{\,3-\upsilon}\ e_j'\ e_i'^{\,\upsilon} = (q^{1/2\, h_{n+1}}\ e_n + q^{-1/2\, h_n}\ e_{n+1})^{3-\upsilon}\ e_{n-1}\ \times$$

$$\times\ (q^{1/2\, h_{n+1}}\ e_n + q^{-1/2\, h_n}\ e_{n+1})^{\upsilon} = 0$$

because there is a_n in e_n and e_{n+1}, and a_n^+ there is only in e_{n-1}.

We proved the following:

Proposition. The relations

$$e_n' = q^{1/2(1-N_n-N_{n+1})}\ a_n\ a_{n+1}^+ + q^{1/2(N_n-N_{n+1})}\ a_n\ a_{n+1}$$

$$(13)\qquad f_n' = a_{n+1}\ a_n^+\ q^{1/2(1-N_n-N_{n+1})} + a_{n+1}^+\ a_n^+\ q^{1/2(N_n-N_{n+1})}$$

$$h_n' = 1-2N_{n+1}$$

$$e_i' = a_i\ a_{i+1}^+,\quad f_i' = a_{i+1}\ a_i^+,\quad H_i' = N_{i+1} - N_i \quad (i=1,\ldots n-1)$$

construct the quantum Clifford algebra representation of $B_q(n)$.

2) D(4) ⇒ G(2). As in (10) the analogs of equations (4) in quantum case have the form:

$$e_1' = q^{1/2(h_2+h_1)}\ e_1 + q^{1/2(h_3-h_1)}\ e_2 + q^{-1/2(h_2+h_1)}\ e_3$$

$$(14)\qquad f_1' = f_1\ q^{1/2(h_2+h_1)} + f_2\ q^{1/2(h_3-h_1)} + f_3\ q^{-1/2(h_2+h_1)}$$

$$h_1' = h_1 + h_2 + h_3$$

$$e_2' = e_0 \qquad f_2' = f_0 \qquad h_2' = h_0$$

Proposition. The elements e_i', f_i', h_i' $(i=1,2)$ satisfy the properties:

$$[\,h_i',e_j'\,] = a_{ij}\ e_j' \qquad [\,h_i',f_j'\,] = -\ a_{ij}\ f_j'$$

(15)

$$[e_i', f_i'] = [h_i']_q$$ Here a_{ij} is Cartan matrix of G(2):

$$a_{ij} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

These equations can be verified in the same way as (11) and (12).

As in the first case, e_i', f_i', h_i' are not the generators of quantum algebra $G_q(2)$ because $[e_1', f_2'] \neq 0$, $[e_2', f_1'] \neq 0$. But in fermion representation (9), in which we replace the indexes:

$$
\begin{array}{llll}
e_1 = a_1 a_2^+ & e_2 = a_4 a_3^+ & e_3 = a_1 a_2 & e_0 = a_3 a_1^+ \\
f_1 = a_2 a_1^+ & f_2 = a_3 a_4^+ & f_3 = a_2^+ a_1^+ & f_0 = a_1 a_3^+ \\
h_1 = N_2 - N_1 & h_2 = N_3 - N_4 & h_3 = 1 - N_1 - N_2 & h_0 = N_1 - N_3
\end{array}
\qquad (16)
$$

it is easy to verify that they vanish.

It remains to verify two quantum Serre equations (6). Again, as in the first case, all terms in (6) vanish. Let us consider for example the case $i=1$, $j=2$. In the term

$$e_1'^{\,4-\nu} e_2' e_1'^{\,\nu} = (q^{(\ )} e_1 + q^{(\ )} e_2 + q^{(\ )} e_3)^{4-\nu} e_0 \times$$

$$\times (q^{(\ )} e_1 + q^{(\ )} e_2 + q^{(\ )} e_3)^{\nu}$$

after opening the brackets all monoms where e_2 enters in power ≥ 2 vanish because there is not a_4^+ in e_i. In other monoms, as we can see from (16), there are three a_1 (in e_1, e_3) and only one a_1^+ (in e_0), so they vanish too. The second equality ($i=2, j=1$) can be checked in the same way.

We proved the

Proposition. Formulas

$$e_1' = q^{1/2(N_2+N_3-N_1-N_4)} a_1 a_2^+ + q^{-1/2(N_2+N_3-N_1-N_4)} a_1 a_2 + + q^{1/2-N_2} a_4 a_3^+ \tag{17}$$

$$f_1' = a_2 a_1^+ q^{1/2(N_2+N_3-N_1-N_4)} + a_2^+ a_1^+ q^{-1/2(N_2+N_3-N_1-N_4)} + + a_3 a_4^+ q^{1/2-N_2}$$

$$h_1' = N_3 - N_4 - 2N_1 + 1$$

$$e_2' = a_3 a_1^+ \quad f_2' = a_1 a_3^+ \quad H_2' = N_1 - N_3$$

generate the quantum Clifford algebra representation of $G_q(2)$.

Note, that we can make the folding D(4)→B(3) in 3 different ways. Here we used the first way of folding. Cases (2) and (3) in fig.2 lead to another fermion representation of $B_q(3)$.

4. FOLDING $A_q(2n-1) \rightarrow C_q(n)$

We will try to carry the quantization of formulas (2) in the following way:

$$e_i' = q^{1/2 h_{2n-i}} e_i + q^{-1/2 h_i} e_{2n-i} \qquad e_n' = e_n$$

$$f_i' = f_i\, q^{1/2 h_{2n-i}} + f_{2n-i}\, q^{-1/2 h_i} \qquad f_n' = f_n \tag{18}$$

$$h_i' = h_i + h_{2n-i} \quad (i=1..n-1) \qquad h_n' = h_n$$

As in the first two cases these formulas are "bad", because $[e_n', f_{n-1}']$ and $[e_{n-1}', f_n']$ don't vanish. Again we take quantum Clifford algebra representation of

$A_q(n)$, which was found in [7] and is

(19) $\qquad e_i = a_i\, a_{i+1}^+ \qquad f_i = a_{i+1}\, a_i^+ \qquad h_i = N_{i+1} - N_i \quad (i=1..n)$

In this case all equalities (5) are satisfied.

It remains to prove the equations (6). Here they are not trivial: the monoms $e_i'^{\,1-a_{ij}-\upsilon}\, e_j'\, e_i'^{\,\upsilon}$ don't vanish. Inserting in (6) into a_{ij} the Cartan matrix of $C(n)$:

$$a_{ij} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & .. & & \\ & & .. & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

we obtain:

(20) $\qquad e_i'^{\,2} e_{i\mp1}' - (q + q^{-1})\, e_i' e_{i\mp1}' e_i' + e_{i\mp1}' e_i'^{\,2} = 0$

for all $(i, i\mp1)$, besides $(i, i+1) = (n-1, n)$

(21) $\qquad \sum_{\upsilon=0}^{3} (-1)^{\upsilon} \left[{3 \atop \upsilon} \right]_q e_{n-1}'^{\,3-\upsilon}\, e_n'\, e_{n-1}'^{\,\upsilon} = 0$

(22) $\qquad e_i'\, e_j' = e_j'\, e_i' \qquad$ (for other i,j)

In the sum (21) all terms vanish as in the previous cases; the equations (22) follows from the analogous ones for $A_q(2n-1)$ (eq.(6) for $a_{ij} = 0$).

We must now prove (20). Substituting (18) in (20) and making the index substitution $i \to 1$, $i+1 \to 2$, $2n-i \to 4$, $2n-i-1 \to 3$ we obtain:

$$\left(q^{1/2 h_4} e_1 + q^{-1/2 h_1} e_4 \right)^2 \left(q^{1/2 h_3} e_2 + q^{-1/2 h_2} e_3 \right) -$$

$$-(q + q^{-1}) \left(q^{1/2 h_4} e_1 + q^{-1/2 h_1} e_4 \right) \left(q^{1/2 h_3} e_2 + q^{-1/2 h_2} e_3 \right) \times$$

$$\times \left(q^{1/2 h_4} e_1 + q^{-1/2 h_1} e_4 \right) + \left(q^{1/2 h_3} e_2 + q^{-1/2 h_2} e_3 \right) \times$$

$$\times \left(q^{1/2 h_4} e_1 + q^{-1/2 h_1} e_4 \right)^2 = 0$$

After opening the brackets, grouping and using (7)

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5. FOLDING $A_q(2n) \to B_q(n)$.

We change formulas (1) in accordance with the previous cases:

$$e_i' = q^{1/2 h_{2n-i+1}} e_i + q^{-1/2 h_i} e_{2n-i+1}$$

(24) $$f_i' = f_i\, q^{1/2 h_{2n-i+1}} + f_{2n-i+1}\, q^{-1/2 h_i}$$

$$h_i' = h_i + h_{2n-i+1} \qquad (i=1..n)$$

Proposition. Formulas (24) satisfy the conditions:

$$[h_i' , e_j'] = a_{ij}\, e_j' \qquad [h_i' , f_j'] = - a_{ij}\, f_j'$$

(25)

$$[e_i' , f_j'] = \delta_{ij}\, [h_j']_q$$

Here a_{ij} is Cartan matrix of B(n).

(20') $$e_i'^2 e_{i\mp1}' - (q + q^{-1})\, e_i' e_{i\mp1}' e_i' + e_{i\mp1}' e_i'^2 = 0 \qquad (i=1..n-1)$$

These equations can be checked in the same way as (20) and (12),(11).

In contrast to the previous case here all equations $[e_i' , f_j'] = \delta_{ij}[h_j']_q$ are valid. But unfortunately e_i, f_i, h_i are not the generators of $B_q(n)$ again because one of the equations (6) (for $i=n$, $j=n-1$) is not valid as it can be checked:

(26) $$e_n'^3 e_{n-1}' + (q^2+1+q^{-2})\, (e_n' e_{n-1}' e_n'^2 - e_n'^2 e_{n-1}' e_n') - e_n' e_{n-1}'^3 \neq 0$$

So we return to fermion representation (19) again, in which it is valid. It can be verified in the same way as (21) was. So, we have the

Proposition. Formulas

$$e_i' = q^{1/2(N_{2n-i+2}-N_{2n-i+1})} a_i \; a_{i+1}^{+} + q^{1/2(N_i-N_{i+1})} a_{2n-i+1} \; a_{2n-i+2}^{+}$$

(27)
$$f_i' = a_{i+1} \; a_i^{+} \; q^{1/2(N_{2n-i+2}-N_{2n-i+1})} + a_{2n-i+2} \; a_{2n-i+1}^{+} \; q^{1/2(N_i-N_{i+1})}$$

$$h_i' = N_{i+1} - N_i + N_{2n-i+2} - N_{2n-i+1}$$

$(i = 1..n)$

are the quantum Clifford algebra representation of $B_q(n)$.

Note, that the folding $A_q(2) \rightarrow B_q(1) = S_q(2,C)$ is valid even without the fremont representation of $A_q(2)$:

(28)
$$e' = q^{1/2 h_2} e_1 + q^{-1/2 h_1} e_2 \quad , f' = f_1 \; q^{1/2 h_2} + f_2 \; q^{-1/2 h_1}$$

$$h' = h_1 + h_2$$

are the homomorphism of q-deformed algebras (but not of Hopf algebras). If we take the quantum boson representation of $A_q(2)$ obtained in [7] we get another boson representation of $Sl_q(2)$:

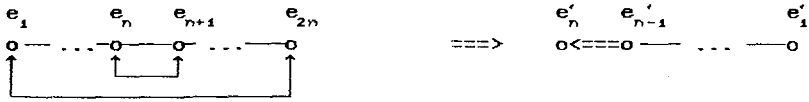
$$e' = q^{1/2(N_3-N_2)} b_1 \; b_2^{+} + q^{1/2(N_1-N_2)} b_2 \; b_3^{+}$$

(29)
$$f' = b_2 \; b_1^{+} \; q^{1/2(N_3-N_2)} + b_3 \; b_2^{+} \; q^{1/2(N_1-N_2)}$$

$$h' = N_3 - N_1$$

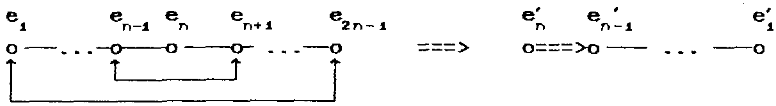
All the considered representations are Hermitian for real q because of the properties: $h_i^{+} = h_i$, $e_i^{+} = f_i$

a)



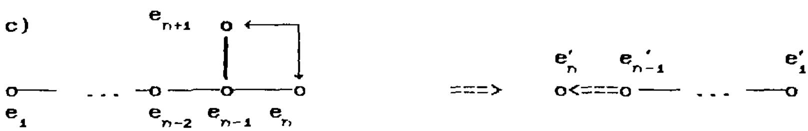
(1) $e'_i = e_i + e_{2n-i+1}$ $\quad f'_i = f_i + f_{2n-i+1}$ $\quad h'_i = h_i + h_{2n-i+1}$

b)



(2) $e'_i = e_i + e_{2n-i}$ $\quad f'_i = f_i + f_{2n-i}$ $\quad h'_i = h_i + h_{2n-i}$

c)

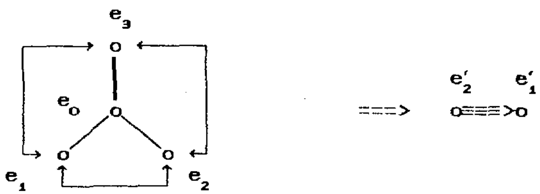


(3)

$e'_n = e_{n+1} + e_n$ $\quad f'_n = f_{n+} + f_n$ $\quad h'_n = h_{n+1} + h_n$

$e'_i = e_i$ $\quad f'_i = f_i$ $\quad h'_i = h_i$ $\quad (i = 1 .. n-1)$

d)



(4)

$e_1 = e_1 + e_2 + e_3$ $\quad f'_1 = f_1 + f_2 + f_3$ $\quad h'_1 = h_1 + h_2 + h_3$

$e'_2 = e_0$ $\quad f'_2 = f_0$ $\quad h'_2 = h_0$

fig.1. Folding of Lee algebras: a) A(2n) → B(n); b) A(2n-1) → → C(n); c) D(n+1) → B(n); d) D(4) → G(2).

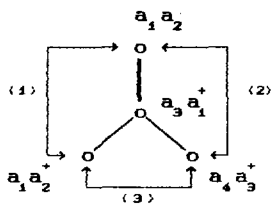


fig.2. Cases of folding D(4) ⇒ B(3).

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The manuskript was reseived in 26 06 1991

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