Preprint YERPHI-1388(18)-92

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE



An.R.KAVALOV, R.L.MKRTCHYAN

# ON A FERMIONIC REALIZATION OF W-TYPE SYMMETRIES



Ереван 1992

Նախնատիպ ԵրՖԻ-1388(18)-92

Ան. Ռ. ԿՈՎԱԼՈՎ, Ռ. Լ. ՄԿՐՏՉՅԱՆ

# W - ՍԻՄԵՏՐԻԱՆԵՐԻ ՖԵՐՄԻՈՆԱՅԻՆ ԻՐԱՑՈՒՄՆԵՐԻ ՄԱՍԻՆ

Դիտարկվում է W պարզագույն հանրահաշիվը 3/2 և 1 սպին ներառնող $\gamma_{i_1 \dots i_{2s}} \psi^{i_1} \dots \psi^{i_{2s}}$ տիպի հոսանքների միջոցով ազատ ֆերմիոնների տեսությունում նրա բոլոր իրացումները գտնելու համար: Պարզվում է, որ Լիի հանրահաշիվների տեսությունում այդ խնդրի լուծումը կարելի է հասցնել որոշակի խնդրի: Բերված է $\gamma$-թենզորների համար լուծումների դասակարգում, որոնք պարզվում է, որ Լիի հանրահաշիվներում կառուցվածքային հաստատունների հետ են կապված: Այդ լուծումը նման է հայտնի բոզոնային կառուցվածքներին, կապված Լիի հանրահաշվի սիմետրիկ նմանակների՝ Ժորդանի հանրահաշիվների հետ:

Երևանի ֆիզիկայի ինստիտուտ

Երևան 1992

W-type symmetries is a common name for a (generally) non-Lie-algebra type symmetries, having a property that commutator of two generators gives some polynomial over the generators (in the case of linear polynomial we come back to the case of the Lie algebras). The first example was found by Zamolodchikov [1], who discovered that two-dimensional conformal theory can have a closed W-type algebra of conserved currents with spins 2 and 3 ($W_3$ algebra), where commutator of spin 3 currents gives a product of two spin 2 currents. In the context of an above-mentioned definition the W-type symmetries were studied in Ref.[2], with the aim of constructing W-gravities - i.e. the theories with localized W-symmetries (see e.g. [2-8] and references therein). Note, that $W_2$-gravity, which is a theory of localized $W_2$ = Virasoro algebra is the usual (and important) Polyakov's 2d-gravity, and that explains the increasing interest to W-gravities .

As a first step for construction of a lagrangian of some W-gravity, or, generally, gauged W-symmetry, one needs some realization of a rigid W-symmetry. The aim of the present paper is to study the free-fermion realizations of some W-type algebras, particularly the simplest one, including spin 1 and spin 3/2 currents. Note that such an algebra (not realizations) was constructed in [10] as a particular case in the study of structure constants of W-algebras with multiplets of spin 1, 3/2 and 2 currents [9,10].

Let us start from the world-sheet action for D free Majorana-Weyl chiral fermions $\psi^i$ of positive chirality:

$$S_0=-(1/2)\int d^2z\ \psi^i\partial_-\psi^i \qquad (1)$$

This theory has a sequence of conserved currents with spins s=1/2, 1, 3/2 ...

$$J_s=-(1/2s)\ \gamma_{i_1\ldots i_{2s}}\ \psi^{i_1}\ldots\psi^{i_{2s}} \qquad (2)$$

generating the symmetries:

$$\delta_s\psi^i=x_s\gamma^i{}_{i_1\ldots i_{2s-1}}\ \psi^{i_1}\ldots\psi^{i_{2s-1}} \qquad (3)$$

of the action $S_0$, with semi-local [3] parameters

$$\partial_- x_s=0 \qquad (4)$$

$\gamma_{i\ldots j}$ are real antisymmetric tensors, with indexes raised and lowered by with the metric - Cronecker symbol - $\delta_{ij}$ (which appears in the action (1)).

The commutator (Poisson bracket) of currents with spin $s_1$ and $s_2$ gives the expression similar to r.h.s. of Eq.(2) with tensor $\gamma$ given by the antisymmetrized contraction of $\gamma$-tensors of currents $J_{s_1}$ and $J_{s_2}$. To have a closed W-type algebra one must have an identity, which permits one to represent that expression as a product of currents from the algebra. Let us materialize this discussion for the case, most extensively studied by us, and the simplest one, where the closed W-type algebra contains only two currents with spins 1 and 3/2:

$$J_1=-(1/2)\gamma_{ij}\psi^i\psi^j \qquad (5)$$

$$J_{3/2}=-(1/3)\gamma_{ijk}\psi^i\psi^j\psi^k \qquad (6)$$

The above-mentioned condition of closure of the algebra in this case means that the following equations have to be satisfied:

$$\gamma_{[i}{}^j\gamma_{pq]j} = \alpha\gamma_{ipq} \qquad , \alpha=0,1 \qquad (7a)$$

$$\gamma_{[ij}{}^k\gamma_{pq]k}= -\varepsilon\gamma_{[ij}\gamma_{pq]} \qquad , \varepsilon=0,\mp 1 \qquad (7b)$$

It was implied in (7), that all possible values of structure constants $\alpha$ and $\varepsilon$ may be brought into the values given above, by choosing the normalization of the (real) tensors $\gamma$. We shall see below that actually the only possibility for $\alpha$ is $\alpha=0$, and we shall not consider the case $\varepsilon=0$ as trivial one. Eq.(7b), which means that commutator of two spin 3/2 currents gives a square of spin 1 current, establishes the W-type nature of this algebra.

As we shall see later, the sign in the r.h.s. of (7b) makes important difference when solving these equations. Also it is worth-wile to mention, that (see below) in the case of non-trivial $\gamma_{ijk}$ the matrix $\gamma_{ij}$ has to be degenerate.

At this point it is relevant to make a comparison with bosonic analogue of above construction [2,3,11], which is a $W_3$ algebra. One considers the free bosonic action

$$S_0=(1/2)\int\partial_+X^i\partial_-X^i \qquad i=1,2,\dots,D \qquad (8)$$

and conserved spin 2 and 3 currents:

$$T_{++}=(1/2)\partial_+X^i\partial_+X^i \tag{9a}$$
$$W_{+++}=(1/3)\partial_+X^i\partial_+X^j\partial_+X^k d_{ijk} \tag{9b}$$

where $d_{ijk}$ is completely symmetric third-rank tensor.

The requirement of closure of this algebra leads to the property that commutator of spin 3 currents gives a square of energy-momentum tensor (spin 2 current), which is equivalent to the following property of $d_{ijk}$ tensor

$$d_{(ij}{}^{s}d_{pq)s}=\sigma\delta_{(ij}\delta_{pq)} \tag{10}$$

So, one may say, that (7b) is an antisymmetric counterpart of (10). It is known [3,11], that solutions of (10) are connected to the Jordan algebras, having a symmetric product of elements. We shall see below that solutions of (7) are connected to the Lie algebras.

Let us prove that in the case of non-trivial spin 3/2 current (i.e. $\gamma_{ijk}\neq 0$) $\alpha$ must be equal to zero. It is actually an almost evident statement, if one realizes, that Eq.(7a) may be presented in the form

$$(\alpha+A)Y=0 \tag{11}$$

where components of vector Y are components of tensor $\gamma_{ijk}$ (enumerated in some arbitrary order) and A is an antisymmetric matrix, constructed from $\gamma_{ij}$. From (11) we deduce that (in the case $Y\neq0$) $\det(\alpha+A)=0$. Skewdiagonalizing A with real eigenvalues $a_\mu$ we present it in a block-diagonal form with 2x2 blocks

$$\begin{pmatrix} 0 & a_\mu \\ -a_\mu & 0 \end{pmatrix}$$

on the main diagonal. So

$$\det(\alpha+A) = \prod_\mu (\alpha^2+a_\mu^2)$$

and can be equal to zero only if $\alpha=0$ and $a_\mu=0$ for some $\mu$. One may show that this last condition leads to some conditions on the (skew) eigenvalues of matrix $\gamma_{ij}$: either the sum of two of them or the sum of three of them must be equal to zero. Note also that the statement $\alpha=0$ is in complete agreement with general form of such algebras, given in [9,10].

Now let us consider the system (7) with $\alpha=0$. Introduce indexes $\mu,\nu,\ldots$ running over the range $0,1,\ldots,D$ , i.e. the range of indexes $i,j\ldots$ plus one more value 0. Introduce also the following tensors: a diagonal metric

$$g_{\mu\nu}=(\epsilon,\ \delta_{ij}), \tag{12}$$

and the completely antisymmetric tensor of structure constants $C_{\mu\nu\rho}$ (indexes of which are raised and lowered with metric $g_{\mu\nu}$) which is constructed from $\gamma_{ijk}$ and $\gamma_{ij}$ :

$$C_{ijk} = \gamma_{ijk} \tag{13a}$$

$$C_{ij0} = \gamma_{ij} \tag{13b}$$

Then relations (7) transform into the familiar form

$$C_{[\mu\nu}{}^{\rho}C_{\lambda]\sigma\rho}=0 \tag{14}$$

of the Jacoby identity for the structure constants of the Lie algebra. The only particular information about that Lie algebra is that it possesses the invariant metric $g_{\mu\nu}$ (the statement that some-second rank tensor with indexes from ad joint representation of Lie algebra is invariant means simply that structure constants of that algebra become completely antisymmetric, when all three indexes are brought on the same level by means of that tensor, as in our case). It is easy to see that, conversely, one may obtain the solution of (7) starting from some Lie algebra possessing invariant tensor $g_{\mu\nu}$ of the form (12) and obtain the solution of (7) by the expressions (10). Introducing generators $X^{\mu}$ we have commutation relations ($X^0=X$)

$$[X_\mu,X_\nu]=C_{\mu\nu}{}^{\lambda}X_\lambda \tag{15a}$$

or

$$[X_i,X_j]=\gamma_{ij}{}^{k}X_k + \gamma_{ij}X \tag{15b}$$

$$[X,X_i]=\gamma_i{}^{s}X_s \tag{15c}$$

So, unexpectedly, we bring the problem of finding the free-fermion realizations of simplest W-algebra with spin 3/2 and spin 1 currents to some problem in the theory of Lie algebra. Here is where the similarity with bosonic case appears: there the problem was reduced [3,11] to the theory of symmetric counterpart of Lie algebras - the Jordan algebras.

So, it remains to classify all Lie algebras with invariant tensor of the form (12), then, noting that SO(D+1) rotations (in the space of indexes $\mu$) keep the form (12) unchanged, but

SO(D) rotations (in the space of indexes i) can be absorbed in the rotation of fermions (this is an invariance of action (1)), one has to find all really different possibilities for structure constants. This problem is not solved up to now in complete form, but we can move a few steps further, in particular, we can find a classes of interesting solutions.

First of all, we note the essential difference between two cases $\varepsilon=1$ and $\varepsilon=-1$. In the first case we deal with Lie algebra, having non-degenerate positive defined bilinear form. It is well-known that such an algebras are classified and turn out to be a direct sum of abelian algebras with semisimple one, so their structure is completely known. In the case $\varepsilon=-1$ invariant bilinear form is not positively defined, and corresponding Lie algebra may have non-trivial radicals.

We give now the examples of both types of solutions. The first example is for $\varepsilon=-1$, and it resembles one of the main cases in the classification of bosonic realization of $W_3$ algebra [11,3]. We give directly the values of $\gamma_{ijk}$ and $\gamma_{ij}$:

$$\gamma_{1i}=0,\ \gamma_{ij}\ (i,j\neq 1)\ \text{arbitrary}, \tag{16a}$$

$$\gamma_{1ij} = \gamma_{ij} \tag{16b}$$

The $\varepsilon=1$ case, as was mentioned above, has to be constructed starting from some semisimple algebra in parameterization where Killing tensor has diagonal form with unit eigenvalues and then choosing one of coordinates as "0" one. Let us give an example for SU(3). Structure constants for SU(3) in Gell-Mann basis $\lambda^{\mu}$ are:

$$C_{123}=2,\ C_{147}=1,\ C_{156}=-1,$$
$$C_{246}=C_{257}=C_{345}=-C_{367}=1,$$
$$C_{458}=C_{678}=\sqrt{3} \qquad (17)$$

The remaining structure constants are equal to zero. Choosing as a zeroth coordinate the coordinate 1 we have for the currents

$$J_1=-2\psi^2\psi^3-\psi^4\psi^7+\psi^5\psi^6 \qquad (18a)$$
$$J_{3/2}=-2\{\psi^2\psi^4\psi^6+\psi^2\psi^5\psi^7+\psi^3\psi^4\psi^5- -\psi^3\psi^6\psi^7+\sqrt{3}\psi^4\psi^5\psi^8+\sqrt{3}\psi^6\psi^7\psi^8\} \qquad (18b)$$

Denoting the corresponding variations of fields $\psi^i$ as $\delta_1$ and $\delta_{3/2}$ (according to (3)) we have

$$\delta_{3/2}J_{3/2}=4\varkappa_{3/2}(J_1)^2 \qquad (19)$$

and zero for all other variations of currents. Taking coordinate 8 as a zeroth one, we obtain for the currents:

$$J_1=-\sqrt{3}(\psi^4\psi^5+\psi^6\psi^7)$$
$$J_{3/2}=-2(2\psi^1\psi^2\psi^3+\psi^1\psi^4\psi^7-\psi^1\psi^5\psi^6+\psi^2\psi^4\psi^5 +\psi^2\psi^5\psi^7+\psi^3\psi^4\psi^5-\psi^3\psi^6\psi^7) \qquad (20)$$

Note, that these two cases differ essentially. In the first case matrix $\gamma_{ij}$ has rank six, but in the second case - four.

The status of the problem of gauging the W-algebra of currents (5), (6) is the following. According to the statement of [2,3], the chiral gauging can be achieved simply by No ether coupling of currents with gauge fields A (boson, spin 1) and $\chi$

(spin 3/2 fermion with chirality opposite to that of $\psi^i$):

$$S= S_0 + \int\{(1/2)\gamma_{ij}\psi^i\psi^j A+(1/3)\gamma_{ijk}\psi^i\psi^j\psi^k \chi\} \qquad (21)$$

with transformations (3) for fields $\psi^i$ with now local parameters $\varkappa_1$, $\varkappa_{3/2}$ (i.e. condition (4) have to be omitted) and transformations

$$\delta A=\partial_-\varkappa_1 + 4\varepsilon\varkappa_{3/2}\chi J_1$$

$$\delta\chi=\partial_-\varkappa_{3/2}$$

for gauge fields.

The interesting and important problem is the calculation of the effective action for the gauge fields A and $\chi$ [3], which is connected to the calculation of the anomalies of corresponding gauge invariances. In the absence of the field $\chi$ the anomaly of the gauge current with spin 1 is well-known - $\partial_- J \approx \partial_+ A$; the second anomaly may be calculated in analogy with Ref.[3].

Finally, the non-chiral gauging in some sense is trivial for these theories, since one has to introduce new fermionic fields with opposite chirality and simply add to the action (21) the similar action with all pluses and minuses interchanged. We are indebted to Al.R.Kavalov for useful discussions.

**References**


[1] A.B. Zamolodchikov, Teor. Mat. Fiz. 65 (1985) 1205

[2] C.M. Hull, Nucl. Phys. B353 (1991) 707-756

[3] C.M. Hull, Phys. Lett. B240 (1990) 110-116

[4] K.Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. B243 (1990) 245-249 249

[5] K.Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B364 (1991) 584-620 620

[6] Y. Matsuo, Phys. Let. B237 (1989) 222

[7] J. Pawelczyk, Phys. Lett. B255 (1991) 330

[8] M. Awada and Z. Qiu, Phys. Lett. B245 (1990) 85

[9] V.G.Knizhnik, Teor. Mat. Fiz. 66 (1986) 102

[10] M.Bershadsky, Phys. Lett. 174B (1986) 285

[11] L.J.Romans, Nucl. Phys. B352 (1991) 829



The manuscript was received 5 November, 1992

Ан. Р. КАВАЛОВ, Р. Л. МКРТЧЯН

О ФЕРМИОННЫХ РЕАЛИЗАЦИЯХ W-СИММЕТРИИ

(на английском языке, перевод авторов)

Редактор А. С. Есин

Технический редактор А. С. Абрамян

Подписано в печать 18/XI -92 Формат 60x84/16
Офсетная печать Уч. изд. л. 0,5 Тираж 100экз., ц. 8р.
Зак. тип. 057 Индекс 3649

Отпечатано в Ереванском физическом институте
Армения, Ереван 375036, ул. Братьев Алиханянов 2.

The address for requests:
Information Department
Yerevan Physics Institute
Alikhanian Brothers 2,
Yerevan, 375036
Armenia,

ИНДЕКС 3649

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ