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THE ANTIFERROMAGNETIC POTTS MODEL EXACT SOLUTION ON THE BETHE LATTICE

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ԱՆԱՆԻԿՅԱՆ Ն.Ս., ՀԱԽԵՅԱՆ Ա.Զ.

ԱՆՏԻՖԵՌՈՄԱԳՆԻՍԱԿԱՆ ՊՈՏՏՍԻ ՄՈԴԵԼ ՃՇԳՐԻՏ ԼՈՒԾՈՒՄ ԲԵՏԵ ՑԱՆՑԻ ՎՐԱ

Բետե ցանցի վրա դիտարկված անտիֆեռոմագնիսական Պոտսի մոդելը արտաքին մագնիսական դաշտում ճշգրիտ լուծված է հաջորդականության հավասարության միջոցով: Դա հնարավոր է դարձնում մոդելի կրիտիկական հատկությունները ուսումնասիրել որպես (x_n) հաջորդականության հատկություններ: Հայտնաբերված է երկրորդ կարգի փուլային անցումների բազմություն:

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It is well known that critical phenomena are closely connected with the behaviour of the non-linear dynamic systems [1]. As an example of such connection we can consider the exact solution of models on the hierarchical lattices. The simplest among them is the Bethe lattice, an infinitely branching tree with all sites having the same coordination number. Recursion relation obtained on that lattice can be identified with the non-linear mapping in the dynamic systems, and properties of the model depends on the behaviour of the iteration sequence $\{x_n\}$ in the thermodynamic limit $n \to \infty$.

In our previous papers we considered the ferromagnetic Potts model, spin-1 Ising model and gauge Potts model on generalized Bethe lattice [2-4]. In all these cases the states of the system are determined from the stable fixed points of the recursive sequence. The presence of more than one stable fixed point means the presence of coexisting phases and first order phase transition taking place when free energies of these phases become equal. The second order transition occurs in some models if these new fixed points arise continuously from old one.

Another picture we obtain for antiferromagnetic models. In this Letter we consider antiferromagnetic Potts model on the Bethe lattice in external magnetic field. On the bipartite lattice we observe the second order phase transition taking place through so-called period doubling bifurcation. Fixed points of the recursive sequence $\{x_n\}$ no longer corresponds to the states of the system and next iteration of recursion function must be considered. Such behaviour one should naturally connect with the division of original lattice into two sublattices.

Antiferromagnetic Potts model [5] receives a great attention recently as a model with interesting an unusual critical properties, such as zero-temperature residual entropy and macroscopically degenerate ground state with algebraic decay of correlations [6-8]. Much in the behaviour of this model still remains in question and using Bethe and other simplified lattices can provide a piece of additional information.

Potts model in the magnetic field is defined by the Hamiltonian:

$$\mathcal{H} = -K\sum_{<ij>}\delta_{\sigma_i,\sigma_j} - h\sum_{i}\delta_{\sigma_i,1} \tag{1}$$

where σ_i takes value 1,2,...Q, first sum goes over all edges and second - over all sites on the lattice. Besides we denote:

$$K = J/kT \qquad h = H/kT$$

and K<0 corresponds to antiferromagnetic case.

Partition function and per site magnetization can be written respectively as:

$$\mathcal{Z} = \sum_{(\sigma)}\exp\left(-\frac{\mathcal{H}}{kT}\right) = \sum_{(\sigma)}\exp\left\{K\sum_{<ij>}\delta_{\sigma_i,\sigma_j} + h\sum_{i}\delta_{\sigma_i,1}\right\} \tag{2}$$

$$M = \langle\delta(\sigma,1)\rangle = Z^{-1}\sum_{(\sigma)}\delta(\sigma,1)\exp\left(-\frac{\mathcal{H}}{kT}\right) \tag{3}$$

Bethe lattice with coordination number $\gamma+1$ may be considered as $\gamma+1$ separate branches connected only in central site. Thus on the lattice, containing n generation partition function of the model may be rewritten in the form:

$$\mathcal{Z} = \sum_{\sigma_0} e^{h\delta(\sigma_0,1)}[g_n(\sigma_0)]^{\gamma+1} \tag{4}$$

where σ_0 is the central spin and $g_n(\sigma_0)$ is the contribution of each lattice branch. The latter is obviously expressed through

$g_{n-1}(\sigma_1)$ i.e. the contribution of the same branch containing n-1 generation and starting from the site belonging to the first generation:

$$g_n(\sigma_0) = \sum_{\sigma_1} \exp\left\{K\delta(\sigma_0,s) + h\delta(\sigma_1,1)\right\}\left[g_{n-1}(\sigma_1)\right]^{\gamma} \qquad (5)$$

Introducing a notation:

$$x_n = \frac{g_n(\sigma\neq 1)}{g_n(\sigma=1)} \qquad (6)$$

the recursion relation (5) can be rewritten in the form:

$$x_n = f(x_{n-1}), \quad \text{where} \quad f(x,K,h) = \frac{e^h + (e^K+Q-2)x^{\gamma}}{e^{K+h} + (Q-1)x^{\gamma}} \qquad (7)$$

x_n has not direct physical meaning but through it one can express magnetization M:

$$M = \frac{e^h}{e^h + (Q-1)x^{\gamma+1}} \qquad (8)$$

and other thermodynamic parameters, since we can say that x_n determine the states of the system.

Fixed points of the iteration sequence $\{x_n\}$ in the thermodynamic limit $n \to \infty$ are the solutions of the equation:

$$x = f(x,K,h) \qquad (9)$$

and they must lie in the interval:

$$\frac{e^K+Q-2}{Q-1} < x < e^{-K} \qquad (10)$$

Unlike the ferromagnetic case (K>0) [2,9] in antiferromagnetic model the recursion function f(x) is monotonously decreasing for all values of K, and h and hence equation (9) has always one and only one solution x_0. This means the absence of coexisting phases and first order phase transitions.

However at low temperature T<T* this single fix point x_0 become unstable:

$$\frac{\partial}{\partial x}\left(f(x,K,h)\right)_{x=x_0} < -1 \qquad (11)$$

Antiferromagnetic Potts model [5] receives a great attention

and so-called period doubling bifurcation take place: the recursive sequence $\{x_n\}$ converges now not to the single fixed point but to the stable 2-cycle $\{x_1;x_2\}$ (fig.1). This fact should be readily explained as an arisement of two-sublattice phase such that x_1 and x_2 determine the states on each sublattice.

To describe the situation now we must consider the second recursion iteration $f^2(x) = f(f(x))$, since the stable solutions of the equation:

$$f^2(x,K,h) = x \tag{12}$$

corresponds to the 2-cycle point x_1, x_2 of $f(x)$. Equation (12) can be expressed as a function $h = h(x,K)$. This function at some temperature $T<T^*$ is shown on fig.2. In the critical behaviour of the system we observe three different cases:

a) At $h>h_c^{(1)}(T)$ and $h<h_c^{(2)}(T)$ equation (12) has one solution coinciding with solution of (9). System is in the disordered paramagnetic phase with no sublattice division and zero staggered magnetization.

b) $h_c^{(1)}(T)< h <h_c^{(2)}(T)$ corresponds to the region of doubling bifurcations. We have the antiferromagnetic ordering, which implies the division on two sublattices and non-zero staggered magnetization:

$$M_s = \frac{1}{2}\Big(M(x_1,K,h) - M(x_2,K,h)\Big) \tag{13}$$

c) For any temperature $T<T^*$ there are two points $(h_c^{(1)},T)$ and $(h_c^{(2)},T)$ (points A and B on fig.2) at which system undergoes the second order phase transition from disordered to two-sublattice ordered phase. Staggered magnetization turn to be non-zero continuously in this point and staggered susceptibility $\chi_s = \partial M_s/\partial H$ diverges.

Having in mind these properties we can describe all critical picture of the model analytically. Thus T^* - the upper bound of temperature at which period doubling and hence ordered phase are possible is given by expression:

$$\exp(J/kT^*) = \frac{1}{2}\left[2-Q + \sqrt{(Q-2)^2 + 4(Q-1)\left(\frac{\gamma-1}{\gamma+1}\right)^2}\,\right] \tag{14}$$

Phase diagram of the model contains the critical line of second order transitions which consists of two branches $h_c^{(1)}(K)$ and $h_c^{(2)}(K)$:

$$h_c^{(1,2)}(K) = \gamma\ln U^{(1,2)} + \ln\left(e^K+Q-2 - (Q-1)U^{(1,2)}\right) - \ln\left(e^K U^{(1,2)}-1\right)$$

and $U^{(1)}$, $U^{(2)}$ are the roots of square equation:

$$U^2 - \left((1+1/\gamma)A + (1-1/\gamma)B\right) U + AB = 0$$

where A and B:

$$A = \frac{e^K+Q-2}{Q-1}, \quad B = e^{-K}$$

coincides with lower and upper bound for x (10)

In conclusion we would like to mention that only one period doubling take place in the model, since second iteration function $f^2(x,K,h)$ appears to be increasing for all values of K and h. Only adding the next nearest neighbour interaction term in the hamiltonian (1) allow one to obtain on the hierarchical lattices the full bifurcation picture including chaos, period-3 windows, strange attractors, etc [10-12]. However we expect that behaviour of the Potts model in this case will be rather complicated.

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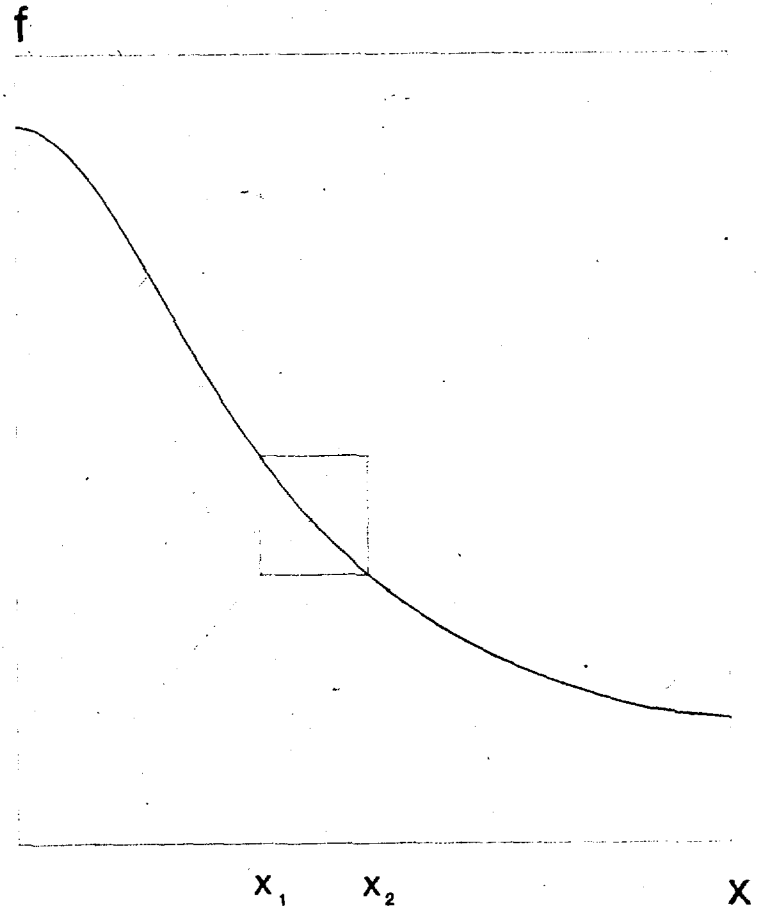

Fig.1 Period doubling bifurcation on the plot of recursion function f(x) at T<T*. Values x_1, x_2 corresponds to stable 2-cycle of iteration sequence $\{x_n\}$

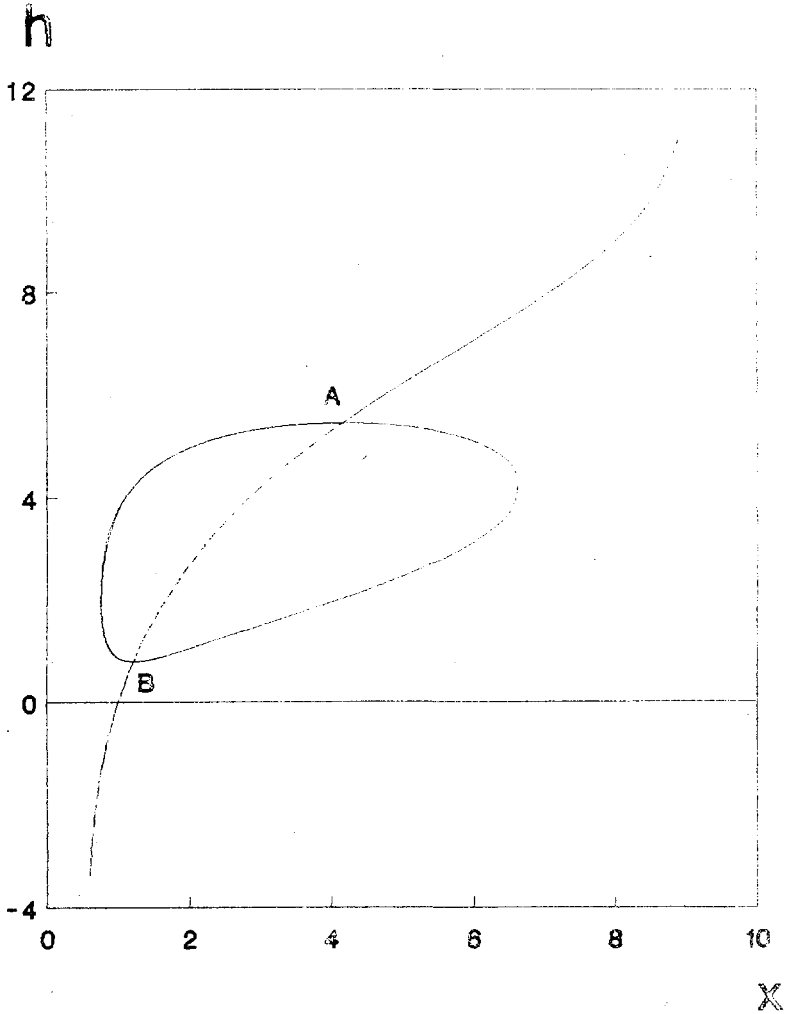

Fig.2 Reduced magnetic field h versus x plot at some $T<T^c$. Solid lines describe the stable states of the system, dashed line corresponds to unstable fixed points. Closed loop on the plot corresponds to the region of period doublings; A and B indicate the points of second order phase transitions from disordered to two-sublattice ordered phase.

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АНТИФЕРРОМАГНИТНАЯ МОДЕЛЬ ПОТТСА. ТОЧНОЕ РЕШЕНИЕ НА РЕШЕТКЕ БЕТЕ.

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THE ANTIFERROMAGNETIC POTTS MODEL
EXACT SOLUTION ON THE BETHE LATTICE

The antiferromagnetic Potts model in the magnetic field is rigorously considered on the Bethe lattice by means of recursion relation. This allow one to study critical properties of the model as properties of iteration sequence $\{x_n\}$ in limit $n \to \infty$. We observe the line of second order phase transition, taking place as a period doubling bifurcations of recursive sequence.

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АНТИФЕРРОМАГНИТНАЯ МОДЕЛЬ ПОТТСА
ТОЧНОЕ РЕШЕНИЕ НА РЕШЕТКЕ БЕТЕ.

Антиферромагнитная модель Поттса во внешнем магнитном поле точно решена на решетке Бете методом рекуррентного уравнения. Это позволяет изучить критические свойства модели как свойства итерационной последовательности $\{x_n\}$ в термодинамическом пределе $n \to \infty$. Мы наблюдаем линию фазовых переходов II рода, происходящих как бифуркация удвоения периода.

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