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ON GAUGE INVARIANCE OF EFFECTIVE ACTION
ON PRECISE SOURCELESS EXTREMAL

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И.А.БАТАЛИН^{*}), Г.К.САВВИДИ

О КАЛИБРОВОЧНОЙ ИНВАРИАНТНОСТИ
ЭФФЕКТИВНОГО ДЕЙСТВИЯ НА ТОЧНОЙ ЭКСТРЕМАЛИ
БЕЗ ИСТОЧНИКОВ

Дано явное доказательство калибровочной инвариантности
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ON PRECISE SOURCELESS EXTREMAL

An explicit proof of gauge invariance of the effective action for fields satisfying the sourceless equations of motion in the Yang-Mills theory is given.

Yerevan Physics Institute

Yerevan 1979

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ON GAUGE INVARIANCE OF EFFECTIVE ACTION

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We shall use, as it was in [1], the form of connected Green functions... the path integral... the gauge group... the classical action $S(\varphi)$ is invariant relative to infinitesimal gauge transformations...

$$\delta \varphi^a(x) = \epsilon^a(x) R^a(\varphi)$$
$$S_1(\varphi) R^a(\varphi) = 0$$

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1. Introduction

An explicit proof of gauge invariance of the effective action for fields satisfying the sourceless equations of motion in non-Abelian gauge theories is given in this work. The gauge invariance of the effective action on a precise extremal by means of the Ward identities was proved in Ref./1/, and in explicit form at one-loop level - in Refs./2, 3/. In the present work we shall generalize the proof /2, 3/ for the cases of arbitrary gauge and more higher orders.

2. Definitions

We shall use, as it was in /4/, two types of condensed indices combining the discrete and continuous space-time coordinates: small Latin indices relate to Lagrange variables of the gauge field φ^i , the small Greek values - to parameters of the gauge group. A classical action $S(\varphi)$ is invariant relative to infinitely small gauge transformations:

$$\delta\varphi^i = R_\alpha^i(\varphi)\delta\xi^\alpha \tag{1}$$

hence

$$S_{,i}(\varphi)R_\alpha^i(\varphi) = 0 \tag{2}$$

$$S_{,ij}(\varphi) R_{\alpha}^i(\varphi) + S_{,i}(\varphi) R_{\alpha,j}^i(\varphi) = 0 \quad (3)$$

etc., where functional derivatives on field are denoted by comma. On a classical extremal

$$S_{,i}(\varphi_0) = 0 \quad (4)$$

these relations have the form:

$$R_{\alpha}^i(\varphi_0) S_{,i}(\varphi_0) = 0 \quad (5)$$

$$R_{\alpha}^i(\varphi_0) S_{,ij}(\varphi_0) = 0 \quad (6)$$

$$R_{\alpha}^i S_{,ijk} + R_{\alpha,k}^i S_{,ij} + R_{\alpha,j}^i S_{,ik} = 0 \quad (7)$$

$$R_{\alpha}^i S_{,ijke} + R_{\alpha,e}^i S_{,ijk} + R_{\alpha,k}^i S_{,ije} + R_{\alpha,j}^i S_{,ike} = 0 \quad (8)$$

$$R_{\alpha,m}^i S_{,ijke} + R_{\alpha,e}^i S_{,ijkm} + R_{\alpha,k}^i S_{,ijem} + \quad (9)$$

$$+ R_{\alpha,j}^i S_{,ikem} = 0$$

and we have taken into account that the action contains a field of no more than fourth power and that $R_{\alpha,jk}^i = 0$

The Lie Algebra of the gauge group G is written in the form:

$$R_{\alpha,j}^i R_{\beta}^j - R_{\beta,j}^i R_{\alpha}^j \equiv R_{\gamma}^i C_{\alpha\beta}^{\gamma} \quad (10)$$

where $C_{\alpha\beta}^{\gamma}$ are structure constants of the group, which satisfy the Jacobi identity:

$$C_{\alpha\epsilon}^{\delta} C_{\beta\gamma}^{\epsilon} + C_{\beta\epsilon}^{\delta} C_{\gamma\alpha}^{\epsilon} + C_{\gamma\epsilon}^{\delta} C_{\alpha\beta}^{\epsilon} = 0 \quad (11)$$

We also assume that:

$$R_{\alpha,i}^i = 0 \quad (12)$$

$$C_{\alpha\beta}^{\beta} = 0 \quad (13)$$

The initial expression for a derivating functional is:

$$\exp\left\{\frac{i}{\hbar} Z\right\} = \quad (14)$$

$$N^{-1} \int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar} [S(\varphi) + \frac{1}{2} \psi^{\alpha} \mathcal{L}_{\alpha\beta} \psi^{\beta} + \mathcal{J}_{\kappa} \varphi^{\kappa}] + S_p \ln [R_{\alpha}^{\kappa} \psi_{,\kappa}^{\beta}]\right\},$$

where $\psi^{\alpha}(\varphi)$ is a calibration function. Using (12) and (13) one can obtain the following expression for variation of the functional at gauge variation:

$$\delta \psi \exp\left\{\frac{i}{\hbar} Z\right\} = \frac{i}{\hbar} \mathcal{J}_{\kappa} R_{\alpha}^{\kappa} \mathcal{D}_{\beta}^{\alpha} \delta \psi^{\beta} \Big|_{\varphi = \frac{\hbar}{i} \frac{\delta}{\delta \mathcal{J}}} \exp\left\{\frac{i}{\hbar} Z\right\}, \quad (15)$$

where

$$R_{\alpha}^{\kappa} \psi_{,\kappa}^{\beta} \mathcal{D}_{\beta}^{\alpha} = -\delta_{\alpha}^{\beta} \quad (16)$$

Let us introduce a derivating functional of Green vertex functions $\Gamma^{(n)}$ (an effective action):

$$\Gamma = Z - \mathcal{J}_{\kappa} \varphi^{\kappa}, \quad (17)$$

where \mathcal{J}_{κ} must be expressed through φ^{κ} by means of the equation

$$\frac{\delta Z}{\delta \mathcal{J}_{\kappa}} = \varphi^{\kappa} \quad (18)$$

Consider a total variation of the functional $\Gamma(\psi)$ at gauge variation

$$\delta_\psi \Gamma = \delta_\psi Z + \frac{\delta Z}{\delta J_K} \delta_\psi J_K - (\delta_\psi J_K) \psi^K \quad (19)$$

The last two terms in (19) compensate each other by virtue of (18) and so we have:

$$\delta_\psi \Gamma(\psi) = \delta_\psi Z(J) \Big|_{J=J(\psi, \psi)} \quad (20)$$

The explicit variation $\delta_\psi Z$ is given by (15) and disappears at $J_K = 0$. Since according to definition (17)

$$\Gamma_{,i}(\psi) = -J_i \quad (21)$$

hence

$$\delta_\psi \Gamma(\psi) = 0, \quad (22)$$

if ψ satisfy the equation

$$\Gamma_{,i}(\psi) = 0 \quad (23)$$

3. Explicit proof of gauge invariance

Now we obtain the expansion $\Gamma(\psi)$ on precise extremal (23) in powers \hbar (i.e. in the number of loops). With this aim we shall present (17) as an expansion in powers \hbar

$$\Gamma(\psi) = \Gamma_0(\psi) + \hbar \Gamma^{(1)}(\psi) + \hbar^2 \Gamma^{(2)}(\psi) + \dots \quad (24)$$

Expanding in powers the solution of equation (23)

$$\psi = \psi_0 + \hbar \psi^{(1)} + \hbar^2 \psi^{(2)} + \dots \quad (25)$$

and substituting it to (23) we have:

$$\Gamma_{0,i}(\psi_0) = 0 \quad (26)$$

$$\Gamma_{0,iK}(\psi_0) \psi^{(1)K} + \Gamma_{,i}^{(1)}(\psi_0) = 0, \quad (27)$$

and so on.

The last expressions allow to express the expansion (25) terms through ψ_0 . For $\Gamma(\psi)$ on the precise extremal expressed through ψ_0 we obtain:

$$\begin{aligned} \Gamma(\psi) \Big|_{(25)} &= \Gamma_0(\psi_0) + \hbar \Gamma^{(1)}(\psi_0) + \quad (28) \\ &+ \hbar^2 \left\{ \Gamma^{(2)}(\psi_0) + \frac{1}{2} \Gamma_{,i}^{(1)}(\psi_0) \Delta^{ij}(\psi_0) \Gamma_{,j}^{(1)}(\psi_0) \right\} + \dots \end{aligned}$$

where according to the definition

$$\Gamma_{0,iK}(\psi_0) \Delta^{Kj}(\psi_0) = -\delta_i^j \quad (29)$$

An unexpected fact is the appearance of the second term in (28) in braces, which is a set of one-particle reducible diagrams. One can be convinced of this fact when substituting (24) as well as the expansion for Z :

$$Z = Z_0 + \hbar Z^{(1)} + \hbar^2 Z^{(2)} + \dots \quad (30)$$

to (17) and (18).

It is of interest to check that each term in (28) does not depend on the gauge. Using the stationary-phase method in applicability to (14) for

the expansion coefficients (28) we obtain:

$$\Gamma_0(\psi_0) = S(\psi_0) + \frac{1}{2} \psi^\alpha \alpha_{\alpha\beta} \psi^\beta, \quad (31)$$

$$\Gamma_0^{(1)}(\psi_0) = \frac{i}{2} S_p \ln \left\{ S_{,ij} + \frac{1}{2} \psi_i^\alpha \alpha_{\alpha\beta} \psi_j^\beta \right\} - i S_p \ln \left\{ R_{,\alpha}^\kappa \psi,^\beta \right\}, \quad (32)$$

$$\Gamma_0^{(2)}(\psi_0) = -\frac{1}{8} S_{,ijkl} \Delta^{ij} \Delta^{\kappa l} \quad (33)$$

$$-\frac{1}{12} S_{,ijk} S_{,emn} \Delta^{ie} \Delta^{jm} \Delta^{\kappa n}$$

$$\frac{1}{2} \mathcal{D}_\alpha^\gamma \mathcal{D}_{\gamma,i}^{-1\alpha} \Delta^{ij} \mathcal{D}_p^\delta \mathcal{D}_{\delta,j}^{-1\beta};$$

$$\frac{1}{2} \Gamma_{,i}^{(1)}(\psi_0) \Delta^{ij}(\psi_0) \Gamma_{,j}^{(1)}(\psi_0) =$$

$$\frac{1}{2} \Gamma_{,i}^{(1)}(\psi_0) \Delta^{ij}(\psi_0) \Gamma_{,j}^{(1)}(\psi_0) =$$

$$-\frac{1}{8} S_{,ijk} S_{,emn} \Delta^{ij} \Delta^{\kappa l} \Delta^{mn}$$

$$-\frac{1}{2} S_{,ijk} \Delta^{ij} \Delta^{\kappa l} \mathcal{D}_\alpha^\beta \mathcal{D}_{\beta,e}^{-1\alpha} \quad (34)$$

$$-\frac{1}{2} \mathcal{D}_\alpha^\beta \mathcal{D}_{\beta,i}^{-1\alpha} \Delta^{ij} \mathcal{D}_p^\delta \mathcal{D}_{\delta,j}^{-1\beta}$$

let us prove that $\Gamma_0(\psi_0)$ is independent of ψ ,

$$\delta\psi \Gamma_0(\psi_0) = \Gamma_{0,i} \delta\psi_0^i + \delta\psi^\alpha \alpha_{\alpha\beta} \psi^\beta = \delta\psi^\alpha \alpha_{\alpha\beta} \psi^\beta(\psi_0) \quad (35)$$

now we write Eq.(28) in the explicit form

$$S_{,i}(\psi_0) + \psi_{,i}^\alpha(\psi_0) \alpha_{\alpha\beta} \psi^\beta(\psi_0) = 0 \quad (36)$$

in virtue of (5) from (36) at convolution with R_γ^i we have:

$$\mathcal{D}_\gamma^{-1\alpha} \alpha_{\alpha\beta} \psi^\beta = 0 \quad (37)$$

i.e.

$$\psi^\beta(\psi_0) = 0 \quad (38)$$

so $\delta\psi \Gamma_0(\psi_0) = 0$. Thus $\Gamma_0(\psi_0)$ does not depend on ψ

Consider $\Gamma_0^{(1)}(\psi_0)$:

$$\delta\psi \Gamma_0^{(1)}(\psi_0) = \Gamma_{,i}^{(1)} \delta\psi \cdot \psi_0^i + \bar{\delta}\psi \Gamma_0^{(1)} \quad (39)$$

where $\bar{\delta}\psi$ means the variation of explicit dependence of $\Gamma_0^{(1)}$ on ψ

For further calculations we need two relations which will be obtained below.

Multiplying Eq.(29) for

$$\left\{ S_{,ij} + \psi_{,i}^\alpha \alpha_{\alpha\beta} \psi_{,j}^\beta \right\} \Delta^{jk} = \delta_i^k \quad (40)$$

by R_γ^i and using (6) we obtain:

$$\alpha_{\alpha\beta} \psi_{,j}^\beta \Delta^{jk} = \mathcal{D}_\alpha^\gamma R_\gamma^k \quad (41)$$

To find the variation of extremal $\delta\psi\psi_0^i$ we vary the equation of motion (36)

$$S_{,ij}\delta\psi\psi_0^j + \psi_{,i}^\alpha \partial_{\alpha\beta}\psi_{,j}^\beta \delta\psi\psi_0^j \quad (42)$$

$$+ \delta\psi_{,i}^\alpha \partial_{\alpha\beta}\psi^\beta + \psi_{,i}^\alpha \partial_{\alpha\beta}\delta\psi^\beta = 0,$$

from which in virtue of (40), (38), (41) we have:

$$\begin{aligned} \delta\psi\psi_0^i &= \Delta^{ij}\delta\psi_{,j}^\alpha \partial_{\alpha\beta}\psi^\beta + \Delta^{ij}\psi_{,j}^\alpha \partial_{\alpha\beta}\delta\psi^\beta \quad (43) \\ &= \Delta^{ij}\psi_{,j}^\alpha \partial_{\alpha\beta}\delta\psi^\beta = R_\alpha^i \mathcal{D}_\beta^\alpha \delta\psi^\beta. \end{aligned}$$

Hence, for explicit variation $\bar{\delta}_\psi \Gamma^{(1)}(\psi_0)$ in virtue of (41) we have:

$$\begin{aligned} \delta\Gamma^{(1)}(\psi_0) &= -\frac{i}{2} \Delta^{ji} \mathcal{Z} \delta\psi_{,i}^\alpha \partial_{\alpha\beta}\psi_{,j}^\beta + i \mathcal{Z}_\alpha^\gamma R_\gamma^i \delta\psi_{,i}^\alpha \quad (44) \\ &= -i \mathcal{D}_\alpha^\gamma R_\gamma^i \delta\psi_{,i}^\alpha + i \mathcal{D}_\alpha^\gamma R_\gamma^i \delta\psi_{,i}^\alpha \equiv 0 \end{aligned}$$

Consider now the first term in (39)

$$\Gamma_{,i}^{(1)} \delta\psi\psi_0^i = \Gamma_{,i}^{(1)} R_\alpha^i \mathcal{D}_\beta^\alpha \delta\psi^\beta \quad (45)$$

for $\Gamma_{,i}^{(1)} R_\alpha^i$ we have:

$$\Gamma_{,k}^{(1)} R_\gamma^k = -\frac{i}{2} \Delta^{ji} S_{,ijk} R_\gamma^k + i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_{\alpha,k}^i R_\gamma^k \quad (46)$$

$$\begin{aligned} &= -\frac{i}{2} \Delta^{ji} [-S_{,ik} R_{\gamma,j}^k - S_{,jk} R_{\gamma,i}^k] + i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_{\alpha,k}^i R_\gamma^k = \\ &= i \Delta^{ji} S_{,ik} R_{\gamma,j}^k + i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_{\alpha,k}^i R_\gamma^k = \\ &= i [-\delta_\kappa^j - \psi_{,\kappa}^\alpha \partial_{\alpha\beta}\psi_{,i}^\beta \Delta^{ij}] R_{\delta,j}^k + i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_{\alpha,\kappa}^i R_\delta^\kappa = \\ &= -i R_{\delta,\kappa}^k - i \mathcal{D}_\beta^\alpha \psi_{,\kappa}^\alpha R_{\delta,j}^k R_\beta^j + i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_{\alpha,\kappa}^i R_\delta^\kappa = \\ &= i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta (R_{\alpha,\kappa}^i R_\delta^\kappa - R_{\delta,\kappa}^i R_\alpha^\kappa) = \\ &= i \mathcal{D}_\beta^\alpha \psi_{,i}^\beta R_\delta^i C_{\alpha\delta}^\beta = -i C_{\delta\gamma}^\beta = 0 \end{aligned}$$

During the transformations we have used the relations (32), (7), (40), (12), (10) and (13).

Consider finally the two-loop contribution to the effective action on the extremal. While calculating the variation one should use the relation

$$\bar{\delta}_\psi \Delta^{j\kappa} = \Delta^{\{ji} \delta\psi_{,i}^\alpha \mathcal{D}_\alpha^\gamma R_\gamma^{\kappa\} \}, \quad (47)$$

where the braces mean symmetrization over indices j and κ . For $\bar{\delta}_\psi \Gamma^{(2)}(\psi_0)$ after protracted calculations we obtain

$$\begin{aligned} \bar{\delta}_\psi \Gamma^{(2)}(\psi_0) &= \frac{1}{2} S_{,mpi} \Delta^{pi} \Delta^{\kappa n} \delta\psi_{,n}^\beta \mathcal{D}_\beta^\alpha R_{\alpha,\kappa}^m \quad (48) \\ &+ \Delta^{\kappa n} \delta\psi_{,n}^\beta \mathcal{D}_\beta^\alpha R_{\alpha,\kappa}^m \mathcal{D}_\beta^\gamma \mathcal{D}_\gamma^{-1} \delta_{\delta,m}^p \end{aligned}$$

and for $\delta\varphi [\frac{1}{2} \Gamma_{,i}^{(1)} \Delta^{ij} \Gamma_{,j}^{(1)}]$ the expression is also obtained, but with an inverse sign. The variation through the extremal can be also calculated in principle by means of the relations obtained in this work.

Let us particularly emphasize the case, when φ_0^i is a covariant-constant field /2/. It is easy to see that $\Gamma_{,i}^{(1)}$ on this field is zero, since the Green gluon $R_\alpha^i \Delta^{ij}/i=j=0$ function in coinciding points $i=j$ is zero. Therefore at calculating the two-loop correction to the effected action /5/ we restricted ourselves with only a first item in braces (28).

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