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A.G.SEDRAKYAN

FERMIONIC DEGREES OF FREEDOM ON A LATTICE: PARTICLES AND STRINGS

ԵՐԵՎԱՆ 1982 ЕРЕВАН

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А.Г.СЕДРАКЯН

ФЕРМИОННЫЕ СТЕПЕНИ СВОБОДЫ НА РЕШЕТКЕ:
ЧАСТИЦЫ И СТРУНЫ

В работе анализируются фермионные степени свободы в двухмерной и трехмерной моделях Изинга. Выражение для фермионного пропагатора в двухмерной модели Изинга, записанного в виде интеграла по траекториям, естественно обобщается на случай фермионной струны в трехмерной модели Изинга и имеет вид суммы по поверхностям. Выведено уравнение для фермионной струны.

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YEREVAN PHYSICS INSTITUTE

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FERMIONIC DEGREES OF FREEDOM ON A LATTICE:
PARTICLES AND STRINGS

Yerevan 1982

The interest in the three-dimensional gauge Z_2 Ising model is caused by the fact that this, simplest gauge system is possibly exactly integrable at least near the critical point [2] . Since in the D=3 Ising model there exists a second order phase transition [3,4], there can be a pass from the lattice to the continuum limit. This fact, together with a natural string structure of gauge theories, gives us a hope of constructing a correct string theory and, hence, a technique to work with gauge theories.

It is known that the two-dimensional D=2 Ising model can be exactly integrated, and its spectrum is of fermionic character [7] . Solving an Ising model, one can write out a propagator of excitation expressed by an integral over trajectories which will satisfy the Dirac equation. These calculations are carried out in the first section of this paper for illustrational purposes.

If we assume that the D=3 Ising model is integrable near the critical point, and its spectrum is composed of a fermion string, we then may naturally generalize the fermion propagator for the case of fermion string, whose expression is presented in section 2. It is an integral over free two-dimensional surfaces. An equation is also obtained which is satisfied by the string propagator. The solution of this equation in a continuum limit seems to give a spectrum of three-dimensional Ising model near the critical point.

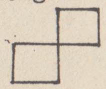
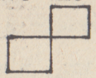
The D=2 Ising model

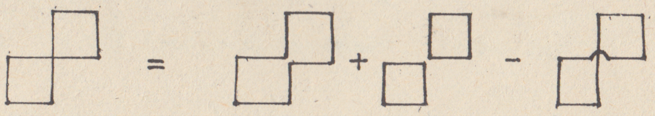
The D=2 Ising model statistical sum has the form

$$Z = \sum_{\sigma_i = \pm 1} e^{-\beta \sum_{i,\mu} \sigma_i \sigma_{i+\mu}} \qquad (1)$$

The value Z may be presented as a sum over all closed paths on the lattice

$$Z = (\mathrm{ch}\beta)^N \sum \lambda^L \qquad (2)$$

$\lambda = \mathrm{th}\,\beta$, and L is the path length. However, this sum includes the contours of the form  with self-intersection. This is not a one-dimensional manifold, i.e., it can't be considered as a function $\vec{x}(\tau)$, which is necessary when writing the Z through an integral over trajectories. In order to have the sum over the functions $\vec{x}(x)$ we must represent  as a sum of trajectories



(3)

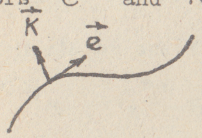
The minus sign near the third term of the right-hand side of (3) leads to the cancellation of its contribution with the first term. In fact the fermionic character of excitations of the D=2 Ising model is stipulated by this fact, as by Pauli's exclusion principle.

The contour in the right-hand side of eq.(3) in the continuum case is included in various homotopical immersion classes $\vec{x}(\tau): S^1 \rightarrow E^2$. These immersion classes are characterized by the cohomological group [8,9]

$$H^1(S^1, \pi_1(SO(2))) = \mathbb{Z} \qquad (4)$$

$\mathbb{Z}$ is the group of integral numbers corresponding to the numbers of self-intersection of contours.

It is not difficult to construct factorized over Z_2 elements H^1. For that purpose consider the element SO(2) formed by a tangent and perpendicular to the contour unit vectors $\vec{e}$ and $\vec{K}$



$$S = \begin{pmatrix} \vec{e} \\ \vec{K} \end{pmatrix} \in SO(2) \qquad (5)$$

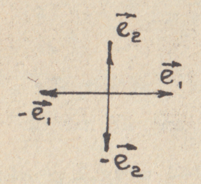
The value

$$\phi(c) = \frac{1}{2}\mathrm{Tr}\exp\left\{\frac{1}{2}\oint_c S\, dS^{-1}\right\} = \frac{1}{2}\mathrm{Tr}\exp\left\{\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\oint d\varphi\right\}. \qquad (6)$$

characterizes the factorized over Z immersion classes of contours S_1 in E^2. Correlating to each contour the multiplier (6) we correctly consider contour signs and in the expression (2) we can sum over free contours:

$$Z = (ch\beta)^N \sum_{\vec{x}(\tau)} \lambda^L \Phi(\vec{x}(\tau)) \qquad (7)$$

It is known [7] that one can obtain the expression (7) by writing a functional integral over Fermi fields. For this purpose let's correlate to each lattice n point a four-component Grassmann field ψ_n and single out its four independent components in the following way:



$$\psi_{\pm e_1}(n) = \frac{1}{2}(1+\gamma_5)(1\pm\vec{e}_1\vec{\gamma})\psi_n$$
$$\psi_{\pm e_2}(n) = \frac{1}{2}(1-\gamma_5)(1\pm\vec{e}_2\vec{\gamma})\psi_n \qquad (8)$$

Here by $1+\vec{e}\vec{\gamma}$ we mean $\begin{pmatrix} 1+\vec{e}\vec{\gamma} & 0 \\ 0 & 1+\vec{e}\vec{\gamma} \end{pmatrix}$ where $\vec{\gamma}$ are Dirac matrices in a two-dimensional space.

Let us write the operator of $\frac{\pi}{2}$ clockwise rotation in the spinor representation SO(2)

$$\Lambda = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & -e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & -e^{-i\frac{\pi}{4}} \end{pmatrix} \qquad (9)$$

Λ^+ is the $-\frac{\pi}{2}$ rotation operator. The unit matrix is the zero angle rotation operator.

Let us write the action for the fermionic field ψ as

$$A = \sum_{n,e} \psi_e^+(n)\psi_e(n) - \lambda \sum_{n,e,\Lambda} \psi_{\Lambda e}^+(n)\Lambda\psi_e(n-e) \qquad (10)$$

It is not difficult to verify that the value

$$W = \int \prod_{n,e} d\psi_e^+(n)\, d\psi_e(n)\, e^{-A} \qquad (11)$$

exactly corresponds to Z^2 in the D=2 Ising model. Let us substitute the expression (8) in eq.(11) and limit ourselves only to the first two components of the field ψ_n because near the critical point only these components contribute in W

The action A in these components takes the form

$$A = \sum_n \psi_n^+\psi_n - \lambda \sum_{n,e} \psi_n^+ [1+\Lambda+\Lambda^+](1+\vec{e}\vec{\gamma})\psi(n-e) =$$
$$= \sum_n \psi_n^+\psi_n - \lambda(1+\sqrt{2}) \sum_{n,e} \psi^+(n)(1+\vec{e}\vec{\gamma})\psi(n-e) \qquad (12)$$

Here $\lambda_c = \frac{1}{\sqrt{2}+1}$ is the critical point of the D=2 Ising model.

Consider the total propagator of fermion propagation from the point $\vec{n}$ to $\vec{n}+\vec{e}_0$, i.e., the value

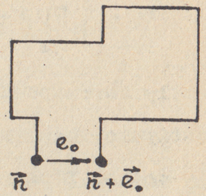
$$\langle \psi_n^+ \psi_{n+e_0} \rangle = \frac{1}{W} \int \prod_n d\psi_n^+ d\psi_n\, \psi_n^+ \psi_{n+e_0}\, e^{-A} \qquad (13)$$

Integrating over Grassmann variables ψ_n we obtain for $\langle \psi_n^+ \psi_{n+e} \rangle$ the expression

$$\langle \psi_n^+ \psi_{n+e} \rangle = \sum_{\text{over all } c} \Phi_c \left(\frac{\lambda}{\lambda_c}\right)(1+\vec{e}\vec{\gamma}) \qquad (14)$$

where the summing is carried out over various contours on the lattice that join the points $\vec{n}$ and $\vec{n}+\vec{e}_0$, and Φ_c is the ordered product of multipliers $(1+\vec{e}\vec{\gamma})$ over the contour C .

Near the critical point, where it is possible to pass on to continuum limit and integrate over free trajectories, one can write instead of the expression (14) the following:



$$<\psi_n^+\psi_{n+e_0}> \sim \sum_L \int_{e_L^2=1} \prod_{i=1}^{L/a} d\vec e_i \prod_{j=1}^{L/a} \frac{\lambda}{\lambda_c}(1+\vec e_j\vec\gamma)\,\delta[(\vec e_0-\sum\vec e_i)a]$$

$$=\sum_{L/a}\int d\vec p\, e^{i\vec p\vec e_0 a}\prod_{j=1}^{L/a}\frac{\lambda}{\lambda_c}\int d\vec e_j\, e^{-i\vec p\vec e_j a}(1+\vec e_j\vec\gamma)=$$

$$=\int d\vec p\, e^{i\vec p\vec e_0 a}\sum_{L/a=0}^{\infty}\left(\frac{\lambda}{\lambda_c}\left(J_0(pa)+iJ_1(pa)\frac{\vec p\vec\gamma}{2p}\right)\right)^{L/a}= \tag{15}$$

$$=\int d\vec p\, e^{i\vec p\vec e_0 a}\frac{1}{1-\frac{\lambda}{\lambda_c}\left[J_0(pa)+iJ_1(pa)\frac{\vec p\vec\gamma}{2p}\right]}\approx$$

$$\approx\int d\vec p\, e^{i\vec p\vec e_0 a}\frac{1}{M+i\frac{\vec p\vec\gamma a}{2}}=Z_0^{-1}(1+ma\vec e_0\gamma)$$

where $d=2$ is the dimension of space, $M=1-\frac{\lambda}{\lambda_c}$ is the departure from the critical point, and $m\sim\frac{1}{Ma}$

The propagator of particles propagation on a finite distance $\vec r$ adopts the form

$$G(\vec r)=\int \mathcal{D}\vec x(\tau)\; P\exp\left\{-m\int_0^{\vec r}\vec\gamma\, d\vec x\right\} \tag{16}$$

where P is the ordered product operator, m is the bare mass of a fermion. In fact, we have shown that the properly regularized expression (16) satisfies the Dirac equation with a renormalized mass $M\sim 1-\frac{\lambda}{\lambda_c}$. Consider now the three-dimensional Ising model.

The D=3 Ising model

It is known [3] that the D=3 spin Ising model is dual to the gauge D=3 Ising model. We shall consider here a gauge D=3 Ising model.

The statistical sum of the gauge D=3 Ising model has the form

$$Z=\sum_{\sigma_{i,\mu}=\pm1} e^{-\beta\sum_{\Box}\sigma_{i,\mu}\sigma_{i+\mu,\nu}\sigma_{i+\nu,\mu}\sigma_{i,\nu}} \tag{17}$$

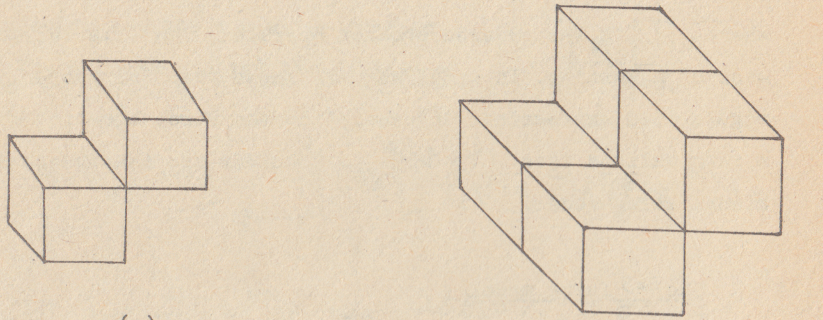
where $\sum_{\Box}$ means the summation over all plaquettes on the lattice, and the spinor variables $\sigma_{i,\mu}$ are correlated to the links of the lattice points i and $i+\mu$.

Expanding the exponent in the expression (17) in series and carrying out the summation over spin variables, one may present Z as a sum over two-dimensional surfaces on the lattice

$$Z=(ch\beta)^N\sum\lambda^A \tag{18}$$

$\lambda = th\beta$, A is the surface area.

However, in this sum are implied surfaces, for example, the ones depicted in fig. 1, that can't be parametrized by the function $\vec{x}(\xi_1, \xi_2)$

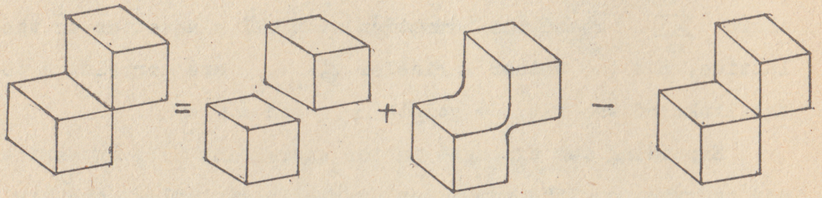


(a) (b)

Fig. 1

In order to represent these surfaces by the function $\vec{x}(\xi_1, \xi_2)$ it is necessary in the same way as in the case of contours to decipher self-intersection lines and correlate to each surface an appropriate sign [5,1] .

For example



It is easy to notice that if the surface has many intersection links, then, deciphering each link in the abovementioned way, we shall always have the required number of cancellations.

In order to pass on to the summation over free surfaces in the formula (18) we must add a multiplier which "feels" the self-intersecting links, correlating to each link a multiplier -1. For this purpose let us consider a class of immersions of the two-dimensional manifolds M_2 into three-dimensional Euclidean space E^3 .

Various immersion classes $\vec{x} : M_2 \to E^3$ are characterized by a cohomological group [8,9]

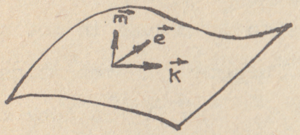
$$H^1(M_2, \pi_1(SO(3)))$$

Let us define the value

$$\phi(c) = \mathrm{Tr}\, P\exp\left\{\oint_c S\, dS^{-1}\right\} \in \frac{H^1}{Z_2} \qquad (19)$$

Here $S \in SO(3)$ is defined in the following way. Consider on the surface the basis formed by three unit vectors $\vec{e}, \vec{k}, \vec{m}$.

$\vec{e}$ and $\vec{k}$ determine the basic directions of the surface in the point ξ , and $\vec{m}$ is the normal in this point. S corresponds to the spinor representation of the element



$$\begin{pmatrix}\vec{e}\\ \vec{k}\\ \vec{m}\end{pmatrix} \in SO(3)$$

In this problem we cannot limit ourselves to immersed surfaces only, because the surfaces of the type (a) and (b) in fig. 1 where the points A and B are singular, should necessarily be taken into account. However, taking into account the fact that the multiplier $\phi(c)$ (eq.(19)) "feels" the self-intersecting links of the surface that the contour C encounters, we can correctly write the sign multiplier taking, as in ref. [1], the product of values of the type $\phi(c)$, determined on various contours surrounding the surface (see fig. 2)

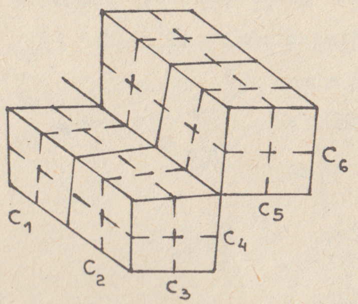


Fig. 2

The sign multiplier in this case takes the form

$$\phi(c_1)\,\phi(c_2)\,\phi(c_3)\,\phi(c_4)\,\phi(c_5)\,\phi(c_6)$$

The expression (19) may be written in another way:

$$\exp\left(SdS^{-1}\right) = 1 + SdS^{-1} =$$

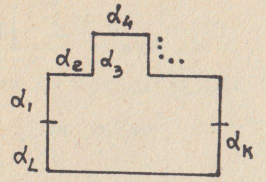
$$= 1 + S(\tau)\left(S^{-1}(\tau+\Delta\tau) - S^{-1}(\tau)\right) = S(\tau)S^{-1}(\tau+\Delta\tau)$$

$$\phi(c) = \mathrm{Tr}\, P \prod_i S(\tau_i)\, S^{-1}(\tau_{i+1}) \qquad (20)$$

Unfortunately, we couldn't manage to introduce in the problem such fermion-type variables, the integration over which would yield the sign multiplier $\prod_c \phi(c)$, as it was done in the $D=2$ Ising model. However, if we assume that such a structure exists, then, naturally generalizing the expression (16), we can write a propagator for it.

Consider [1,6] the contour C of the length L and

the wave function * $\psi_{\alpha_1 \ldots \alpha_L}(c)$ with 2^L degrees of freedom



defined on the contour. The motion of this string is reduced to the addition of the plaquette $\sigma_{\mu\nu}$ to the contour C. The probability of the fact that somewhere the contour is deformed on the plaquette $\sigma_{\mu\nu}$ is defined by the string propagator.

The natural generalization of the formula (15) in this case is

$$\langle \psi_{\alpha_1\alpha_2\alpha_3\alpha_4}(\square) = \alpha_1 \overset{\alpha_2}{\underset{\alpha_4}{\boxed{+}}} \alpha_3 \rangle =$$

$$= \left(1\otimes 1 + m\,\tau_\alpha \otimes \tau_\beta\, \delta\sigma_{\alpha\beta}\right)_{\alpha_1\alpha_3,\,\alpha_2\alpha_4} = \qquad (21)$$

$$= S\otimes S\left(1 + m\,\tau_1\otimes\tau_2\,\delta\sigma_{12}\right) S^{-1}\otimes S^{-1}$$

where $\vec{\tau}$ are Pauli matrices, S are the operators defined in (20),

$$\delta\sigma_{\alpha\beta} = \delta x_\alpha\, dx_\beta - \delta x_\beta\, dx_\alpha$$

and m is the bare string tension.

In order to write a string propagator for any contour C let us introduce, following ref. [1], the geometrical multiplication of tensors $\psi_{\alpha_1 \ldots \alpha_L}(c)$. Indices $\alpha_1, \ldots \alpha_L$ correspond to fermionic degrees of freedom on every link of the lattice on the contour C.

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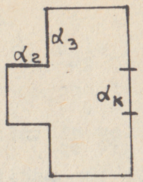
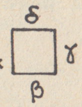
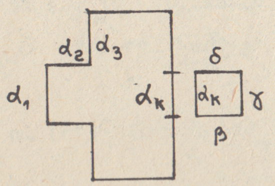
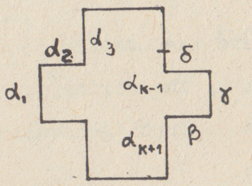
* Such constructions were first introduced by A.M.Polyakov.

$\psi_{\alpha_1 \cdots \alpha_L}(c) =$ 

$\psi_{\alpha_K \beta \gamma \delta} =$ 



$=$  $=$

$$= \sum_{\alpha_K} \psi_{\alpha_1 \cdots \alpha_K \cdots \alpha_L} \cdot \psi_{\alpha_K \beta \gamma \delta} =$$

$$= \psi_{\alpha_1 \cdots \alpha_{K-1}\, \beta \gamma \delta\, \alpha_{K+1} \cdots \alpha_L}$$

Then the string propagator is defined as

$$\langle \psi_{\alpha_1 \cdots \alpha_L}(c) \rangle = \sum_{\substack{\text{over all surfaces with the} \\ \text{boundary } C}}{}' R \prod_{\Box} \left(1 + m\, \tau_\alpha \otimes \tau_\beta\, \delta \sigma_{\alpha\beta}\right) = \tag{22}$$

$$= \sum R \exp\left\{ m\, \tau_\alpha \otimes \tau_\beta \int d\sigma_{\alpha\beta} \right\}$$

where R is the operator of surface ordering of geometrical product over plaquettes.

It is not difficult to derive an equation which is satisfied by the value

$$G(c) = \langle \psi_{\alpha_1 \cdots \alpha_L}(c) \rangle$$

Symbolically this equation has the form

$$G(c) = \sum_{\substack{\text{over all} \\ \text{along the con-} \\ \text{tour length}}} G'(c + \sigma_{\mu\nu})\left(1 + m\, \tau_\mu \otimes \tau_\nu\, \delta \sigma_{\mu\nu}\right) \tag{23}$$

Let us expand $G(c + \sigma_{\mu\nu})$ in Taylor series

$$G(c + \sigma_{\mu\nu}) = G(c) + \sigma_{\mu\nu} \frac{\delta G}{\delta \sigma_{\mu\nu}} + \cdots \tag{24}$$

and substitute it in (23)

$$G(c) = \sum \Big\{ G(c) + G(c)\, m a^2\, \tau_\mu \otimes \tau_\nu +$$

$$+ \sigma_{\mu\nu} \frac{\delta G}{\delta \sigma_{\mu\nu}} + m \frac{\delta G}{\delta \sigma_{\mu\nu}} \sigma_{\mu\nu} \left(\tau_k \otimes \tau_e\, \delta \sigma_{ke}\right) \Big\} =$$

$$= \sum G(c) + m a^4 \sum \frac{\delta G}{\delta \sigma_{\mu\nu}(s)} \left(\tau_\mu \otimes \tau_\nu\right)$$

Thus, we obtain equation

$$\sum_{\substack{\text{along the con-} \\ \text{tour length}}} \frac{\delta G}{\delta \sigma_{\mu\nu}(s)} \left(\tau_\mu \otimes \tau_\nu\right) = M \sum_{\substack{\text{along the con-} \\ \text{tour length}}} G(c) \tag{25}$$

In (25) M is proportional to the renormalized string tension and characterizes the departure from the critical point.

The expression (22) for the propagator as well as eq.(25) make sense on the lattice, and their continuum limit is still obscure. There is a lot of work in this direction. It is possible that the functions $\psi_{\alpha_1 \cdots \alpha_L}(C)$ in the continuum limit should be understood as a spinor representation of Kac-Moody algebra.

In conclusion I wish to thank A.M.Polyakov for numerous discussions as well as H.M.Asatryan and S.G.Matinyan for their remarks.

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ФЕРМИОННЫЕ СТЕПЕНИ СВОБОДЫ НА РЕШЕТКЕ:
ЧАСТИЦЫ И СТРУНЫ

(на английском языке, перевод Л.Н.Багдасаряна)

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