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G.V.GRIGORYAN, R.P.GRIGORYAN, I.V.TYUTIN

ON THE EQUIVALENCE OF HAMILTONIAN AND LAGRANGIAN APPROACHES IN QUANTUM FIELD THEORY

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Г.В.ГРИГОРЯН, Р.П.ГРИГОРЯН, И.В.ТЮТИН х)

К ВОПРОСУ ОБ ЭКВИВАЛЕНТНОСТИ ГАМИЛЬТОНОВА И ЛАГРАНЖЕВА ПОДХОДОВ В КВАНТОВОЙ ТЕОРИИ ПОЛЯ

Обсуждается эквивалентность лагранжева и гамильтонового подходов в квантовой теории поля для случая локальных лоренц-инвариантных лагранжианов без связей. Доказана их эквивалентность при использовании подходящей регуляризации (размерной регуляризации). Эквивалентность обоих подходов доказана также для класса неабелевых калибровочных теорий вида $L(G^2_{\mu\nu})$.

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G.V.GRIGORYAN, R.P.GRIGORYAN, I.V.TYUTIN[*]

ON THE EQUIVALENCE OF HAMILTONIAN AND LAGRANGIAN APPROACHES IN QUANTUM FIELD THEORY

The equivalence of the Lagrangian and Hamiltonian approaches in quantum field theory for the local Lorentz-invariant lagrangian without constraints is discussed. The equivalence is proved using the suitable regularization (dimensional regularization). The equivalence of the two approaches is proved also for the nonabelian gauge theories of the type $L\left(G_{\mu\nu}^{a}\right)$

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1. As it is known, the S matrix in quantum field theory can be described both in Lagrangian and Hamiltonian formalisms (in the presence of constraints the "Hamiltonian formalism" is the Dirac's method of quantization of systems with constraints) [1-3] . Each of these approaches has its advantages. The Hamiltonian approach is suitable for the construction of a theory which is manifestly unitary in the physical subspace, whereas the Lagrangian formalism ensures an evident Lorentz covariance. Besides, all perturbation theory calculations are usually carried out in the Lagrangian formalism. In all theories dealt with in physics so far these two approaches lead to the same S -matrix. However a general statement about the coincidence of the S matrices in these approaches for arbitrary theories does not exist.

To clarify the problem consider the generating functional for the theory without constraints in Hamiltonian formalism

$$Z_H(J) = \int d\phi \, d\Pi \exp\left\{ i \int dx \left[\Pi\dot{\phi} - \tilde{H}(\Pi,\phi) + J\phi \right] \right\} \tag{1}$$

Let us transform it singling out in the exponent the lagrangian using the relation $\tilde{H}(\Pi_1,\phi) = \Pi_1\dot{\phi} - \tilde{L}(\phi,\dot{\phi})$, where Π_1 is defined from the equation $\dot{\phi} = \frac{\partial H(\Pi_1 \phi)}{\partial \Pi_1}$. Changing the integration variable $\Pi = \Pi_1 + \Pi'$ we represent $Z(J)$ in the form

$$Z_H(J) = \int d\phi \, \exp\Big\{ i \int dx \big[\tilde{L} + J\phi \big] \Big\} \Delta(\phi) \tag{2}$$

where

$$\Delta(\phi) = \int d\Pi' \exp\Big\{ -i \int dx \big[\tilde{H}(\Pi_1 + \Pi', \phi) - \Pi \frac{\partial \tilde{H}(\Pi_1 \phi)}{\partial \Pi_1} - \tilde{H}(\Pi, \phi) \Big\}$$

It is clear that the two formalisms are equivalent if $\Delta(\phi) \sim 1$

In this paper the equivalence of the Lagrangian and Hamiltonian formalisms is proved for the local Lorentz -invariant lagrangian without gauge symmetries (Sec.2) and for the gauge theories described by the lagrangian $L = L(G^2_{\mu\nu})$ (Sec.3).

2. Let us consider theories without constraints which are described by local Lorentz-invariant lagrangians $L = L(\partial_\mu \varphi, \varphi)$, $(\mu = 0,1,2,3)$ with terms quadratic over the fields in the usual form.

Then by definition

$$\Pi(x) = \frac{\delta L}{\delta \dot{\varphi}(x)} \quad , \qquad H = \int \Pi(x) \dot{\varphi}(x) dx - L = \int dx \, \tilde{H} \tag{3}$$

The generating functional in the Hamiltonian formalism is

$$Z(J) = \int d\phi \, d\Pi \, \exp\Big\{ i \int dx \big[\Pi \dot{\phi} - \tilde{H} + J\phi \big] \Big\} \tag{4}$$

Turning now in (4) from the integration over Π to an integration over Q related to Π by the equation

$$\Pi(\phi(x), Q(x)) = \frac{\delta L(Q, \partial_\kappa\phi, \phi)}{\delta Q(x)}$$

with a subsequent shift of the integration variable $Q \to \dot\phi + Q$ come to

$$Z(J) = \int d\phi\, dQ \det\left|\frac{\delta\Pi(x)}{\delta Q(y)}\right| \exp\left\{i\int dx\left[\Pi(Q,\phi)\dot\phi - \tilde H(\Pi(Q,\phi),\phi) + J\phi\right]\right\} = \tag{5}$$

$$= \int d\phi\, dQ \det\left|\frac{\delta^2 L(\phi, \dot\phi + Q)}{\delta Q(x)\delta Q(y)}\right| \exp\left\{i\int dx\left[\Pi(\dot\phi + Q,\phi)\dot\phi - \tilde H(\Pi(Q+\dot\phi,\phi),\phi) + J\phi\right]\right\} =$$

$$= \int d\phi \exp\left\{i\int dx\left[\tilde L(x) + J\phi\right]\right\} \Delta(\phi)$$

where

$$\Delta(\phi) = \int dQ \det\left|\frac{\delta^2 L(\dot\phi + Q, \phi)}{\delta Q(x)\,\delta Q(y)}\right| \times \tag{6}$$

$$\times \exp\left\{i\int dx\left[\tilde L((\dot\phi + Q),\phi) - \tilde L(\dot\phi,\phi) - \frac{\partial \tilde L(\phi + Q,\phi)}{\partial Q} Q\right]\right\}$$

We shall prove now that under suitably chosen regularization of the theory $\Delta(\phi) = 1$ (for definiteness we use dimensional regularization scheme). Let us consider the first factor of the integrand. Owing to the supposed locality of the lagrangian

$$\Pi(x) = \frac{\delta L(\dot{\phi}, \partial_\kappa \phi, \phi}{\partial \phi(x)} = \dot{\phi}(x) + \tilde{\Pi}\left(\dot{\phi}(x), \partial_\kappa \phi(x), \phi(x)\right). \tag{7}$$

Then

$$\frac{\delta^2 L(\dot{\phi} + Q, \phi}{\delta Q(x)\, \delta Q(y)} = \frac{\delta \Pi(\dot{\phi})(x) + Q(x), \partial_\kappa \phi(x), \phi(x))}{\delta Q(y)} =$$

$$= \delta(x-y) + \delta(x-y) \frac{\partial \tilde{\Pi}(\dot{\phi}(x) + Q(x), \partial_\kappa \phi(x), \phi(x))}{\partial Q(x)} \tag{8}$$

Hence

$$\det\left| \frac{\delta^2 L}{\delta Q(x)\, \delta Q(y)} \right| = \exp\left\{ Sp \cdot \ln\left[\delta(x-y) + \delta(x-y) \frac{\partial \tilde{\Pi}(x)}{\delta Q(x)} \right] \right\} =$$

$$= \exp\left\{ Sp\left[\delta(x-y) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\partial \tilde{\Pi}(x)}{\partial Q(x)} \right)^n \right] \right\} = \tag{9}$$

$$= \exp\left\{ \delta(0) \ln\left(1 + \frac{\partial \tilde{\Pi}(x)}{\partial Q(x)}\right) \right\} = 1$$

since in the dimensional regularization scheme $\delta^{(n)}(0) = 0$.

Consider now the second factor of the integrand. The power of the exponent (let us denote it by Λ) may be rewritten in the form

$$\Lambda(\phi, Q, \phi) = \tilde{L}(\dot{\phi} + Q, \phi) - \tilde{L}(\dot{\phi}, \phi) - \frac{\partial \tilde{L}(\dot{\phi} + Q, \phi)}{\partial Q} Q = \tag{10}$$

$$= Q(x)\, \Lambda_1(\dot{\phi}(x), \phi(x))\, Q(x) + O(Q^3)$$

(in writing down the latter expression the locality of the lagrangian was taken into account; Λ_1 is a local function of $\dot{\phi}$ and ϕ without derivatives due to the supposed standard form of kinetic term in L). Thus

$$< Q(x)\, Q(y) > \sim \delta^{(n)}(x - y) \tag{11}$$

The evaluation of the integral over Q by perturbation methods reduces to a calculation of the vacuum expectations of the type $< Q^m >$ which due to (11) are proportional to $\delta^n(0)$ and within the dimensional regularization scheme are equal to zero.

Therefore $\Delta(\phi) = 1$ and the Lagrangian and Hamiltonian formalisms are equivalent.

3. Now consider a theory described by a lagrangian

$$L = \int dx\, \tilde{L}\left[(G^a_{\mu\nu})^2\right] = \int dx\, \tilde{L}\left[G^a_{0i},\ G^a_{ik}\right], \tag{12}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu; \quad \mu, \nu = 0, 1, 2, 3;$$
$$i, k = 1, 2, 3$$

To construct the hamiltonian of the system we must find the canonical momenta

$$\pi^{a,o}(x) = \frac{\delta L}{\delta \dot{A}_o(x)} = 0 \tag{13}$$

$$\pi^{a,i}(x) = \frac{\delta L}{\delta \dot{A}_i^a(x)} = \frac{\delta L}{\delta G_{oi}^a(x)} = \pi^{a,i}\left(G_{oi}^b(x),\, G_{ik}^b(x)\right) \tag{14}$$

Eqs.(13) represent primary constraints and consequently this system is a constrained one and must be quantized by Dirac's method [1] . To elaborate the hamiltonian we have to make use of (14) to express $G_{oi}^a(x)$ in terms of $\pi^{i,a}$ and G_{ik}^a . Then the hamiltonian reads

$$\begin{aligned} H &= \int dx\, \pi^{\mu,a} \dot{A}_\mu^a(x) - L = \int dx\, \pi^{a,i}(x) \dot{A}_i^a(x) - L = \\ &= \int dx\, \pi^{a,i}(x) G_{oi}^a(x) + \int dx\, \pi^{a,i}(x) \nabla_i^{ab} A_o^b(x) - L\left(G_{oi}(\pi^j, G_{je}), G_{ik}\right) = \\ &= \int dx\, \pi^{a,i}(x) G_{oi}^a(x) - L\left(G_{oi}(\pi^j, G_{je}), G_{ik}\right) - \int dx\, A_o^b(x) \nabla_i^{ba} \pi^{a,i}(x) = \\ &= H_1 - \int dx\, A_o^b(x) \nabla_i^{ba} \pi^{a,i}(x). \end{aligned} \tag{15}$$

Following Dirac we incorporate the constraints in the hamiltonian using the Lagrange multipliers $H^* = H + \int \lambda^a(x)\, \pi^{a,o}(x)\, dx$. Then the compatibility condition

$$\left\{H^*,\ \pi^{a,o}(x)\right\} = 0 \tag{16}$$

leads to a secondary constraint

$$\nabla_i^{ba}\,\Pi^{a,i}(x) = 0 \tag{17}$$

Here $\{\cdots\cdots\}$ are Poisson brackets defined by

$$\left\{f(z), g(t)\right\} = \int dx \left[\frac{\delta f(z)}{\delta \Pi^{\mu,a}(x)}\,\frac{\delta g(t)}{\delta A^a_\mu(x)} - \frac{\delta f(z)}{\delta A^a_\mu(x)}\,\frac{\delta g(t)}{\delta \Pi^{\mu,a}(x)}\right]$$

Now we must find out whether other secondary constraints exist. To achieve this end we must compute the Poisson bracket

$$\left\{\nabla_i^{ab}\Pi^{b,i}(x), H^*\right\} = \left\{\nabla_i^{ab}\Pi^{b,i}(x), H_1\right\} -$$

$$- \int A_0^c(y)\,dy \left\{\nabla_i^{ab}\Pi^{b,i}(x),\ \nabla_j^{cd}\Pi^{d,j}(y)\right\}.$$

In obtaining the latter we used the equation

$$\left\{\nabla_i^{ab}\Pi^{b,i}(x),\ \Pi^{0,c}(y)\right\} = 0$$

It is easy to show that the following relations hold

$$\left\{\nabla_i^{ab}\Pi^{i,b}(x), \nabla_k^{cd}\Pi^{k,d}(y)\right\} = g\,\delta(x-y)\,f^{abc}\,\nabla_j^{bd}\Pi^{d,j}(x) \approx 0 \tag{18}$$

$$\left\{\nabla_i^{ab}\,\Pi^{b,i}(x), H_1\right\} = 0 \tag{19}$$

and hence

$$\left\{\nabla_i^{ab}\Pi^{b,i}(x), H^*\right\} = -\,g\,A_0^c(x)\,f^{abc}\,\nabla_j^{bd}\Pi^{d,j}(x) \approx 0 \tag{20}$$

where $\approx$ denotes that the equation holds on constraints.

Note that eqs.(19) and (20) are manifestations of the gauge-invariance of H_1 and H^*, respectively.

The equality (20) means that additional secondary constraints do not exist. Let us now introduce an auxiliary condition

$$\partial_i A_i^a(x) = 0 \tag{21}$$

Following the general formalism this constraint must be added to the hamiltonian using the Lagrange multiplier

$$H_2 = H^* + \int dx\, \mu^a(x)\, \partial_i A_i^a(x) \tag{22}$$

and Poisson brackets between this hamiltonian and the constraints (13), (17) and (21) must be computed. This leads to the following relations:

$$\left\{ \pi^{a,0}(x), H_2 \right\} = \nabla_i^{ab} \pi^{b,i} \approx 0 \tag{23}$$

$$\left\{ \nabla_i^{ab} \pi^{b,i}(x), H_2 \right\} = -g A_0^c(x) f^{abc} \nabla_i^{bd} \pi^{d,i}(x) + \nabla_i^{ab} \partial_i \mu^b(x) \approx 0 \tag{24}$$

$$\left\{ \partial_i A_i^a, H_2 \right\} = -\partial_i G_{0i}^a(x) - \partial_i \nabla_i^{ab} A_0^b(x) \approx 0 \tag{25}$$

The relation (23) holds due to the (17), while constraints (17) and (21) reduce (24) to the equation

$$\left(\Delta \delta^{ac} + g f^{abc} A_i^b\, \partial_i \right) \mu^c(x) = 0 \tag{26}$$

from which it follows that $\mu^c(x) = 0$.

Finally, (25) leads to a new constraint

$$A_0^a - A_0^a\left(A_i^b(x),\, \pi^{b,i}(x)\right) = 0 \tag{27}$$

The compatibility condition for this new constraint

$$\left\{A_0^a - A_0^a(A_i^b(x), \pi^{b,i}(x)), H_2\right\} = \lambda^a - \left\{A_0^a(A_i^b(x), \pi^{b,i}(x)), H_2\right\} \approx 0 \tag{28}$$

is an equation from which the Lagrange multiplier $\lambda^a(x)$ is defined.

Thus all Lagrange multipliers $\lambda^a(x)$, $\mu^a(x)$, are found and the total set of constraints ϕ_n^a is

$$(\Phi_n^a) = \left(\pi^{a,0}(x), \nabla_i^{ab}\pi^{b,i}(x), A_0^a - A_0^a(\pi^{b,j}(x), A_j^b(x)), \partial_j A_j^a(x)\right) = \tag{29}$$

$$= \left(\Phi_1, \Phi_2, \Phi_3, \Phi_4\right)$$

To write down the generating functional in the Hamiltonian formalism we must compute the determinant of the matrix $\left[\{\phi_n^a, \phi_m^b\}\right]$. The matrix is readily calculated and has the following form:

$$\left|\{\Phi_n^a, \Phi_m^b\}\right| = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & f^{ab}\nabla_i^{cd}\pi^{d,i} & \{\phi_2^a, \phi_3^b\} & \partial_i \nabla_i^{ab} \\ 1 & -\{\phi_2^a, \phi_3^b\} & \{\phi_3^a, \phi_3^b\} & \{\phi_3^a, \phi_4^b\} \\ 0 & -\partial_i \nabla_i^{ab} & -\{\phi_3^a, \phi_4^b\} & 0 \end{vmatrix} \tag{30}$$

The explicit expressions of the Poisson brackets $\{\phi_2^a, \phi_3^b\}$, $\{\phi_3^a, \phi_3^b\}$, $\{\phi_3^a, \phi_4^b\}$ are not presented here because they do not contribute to the determinant.

From (30) we find the determinant

$$\left|\det\left[\left\{\phi_n^a, \phi_m^b\right\}\right]\right| = \det\left(\partial_i \nabla_i^{ab}\right)^2 = \left[\det \partial_i \nabla_i^{ab}\right]^2 \tag{31}$$

Now we can write down the generating functional in the Hamiltonian formalism 2-4

$$Z_H = \int dA_\mu^a d\pi_\mu^a \sqrt{\left|\det\left\{\phi_n^a \phi_m^b\right\}\right|} \prod_n \delta(\phi_n) \exp\left\{i\int dx \pi^{a,\mu}(x) \dot{A}_\mu(x) - H\right\} =$$

$$= \int dA_i^a \delta(\partial_i A_i^a) \det\left(\partial_j \nabla_j^{ab}\right) \int dA_0^a d\pi_0^a \delta\left(A_0^a - A_0^a\left(A_i^b(x), \pi_i^b(x)\right)\right) \delta\left(\pi_0^a\right) \times \tag{32}$$

$$\times \int d\pi_i^a \delta\left(\nabla_j^{ab} \pi^{bj}\right) \exp\left\{i \int \pi^{a,i}(x) \dot{A}_i^a(x) dx - H_1\right\}$$

Integrals over A_0^a and π_0^a are equal to unity. Therefore

$$Z_H = \int dA_i^a \delta\left(\partial_i A_i^a\right) \det\left(\partial_i \nabla_i^{ab}\right) \times$$
$$\times \int d\pi_i^a \delta\left(\nabla_i^{ab} \pi^{b,i}\right) \exp\left\{i \int \pi^{a,i}(x) \dot{A}_i^a(x) dx - H_1\right\} \tag{33}$$

Substituting $\delta\left(\nabla_i^{ab} \pi^{b,i}\right)$ in (32) by

$$\delta\left(\nabla_i^{ab} \pi^{b,i}\right) = \int d\lambda^a \exp\left\{-i \int \nabla_i^{ab} \pi^{b,i}(x) \lambda^a(x) dx\right\} \tag{34}$$

we find that $Z_H = \int dA_i^a \delta\left(\partial_i A_i^a\right) \det\left(\partial_i \nabla_i^{ab}\right) \times$

$$\times \int d\pi_i^a d\lambda^a \exp\left\{i \int \pi^{b,i} \dot{A}_i^b(x) - H_1 - \int \nabla_i^{ab} \pi^{b,i} \lambda^a(x) dx\right\} \tag{35}$$

If we now identify $\lambda^a(x)$ with $A_0^a(x)$ and change the integration over π_i^a by the one over Q_i^a related to $\pi^{a,i}$ by

$$\pi^{a,i} = \frac{\delta L(Q_j^b,\ A_j^b}{\delta Q_i^a} \tag{36}$$

(again, as it was in Sec.2, the jacobian $\left|\delta^2 L/\delta Q_i \delta Q_j\right| = 1$), then after shifting the integration variable $Q_i^a \to Q_i^a - \nabla_i^{ab} A_0^b + G_{0i}^a$ we get

$$Z_H = \int dA_i^a\, dA_0^a\, \delta(\partial_i A_i^a) \det(\partial_i \nabla_i^{ab}) e^{iL(G_{0i}^a,\, G_{ik}^a)} \times$$

$$\times \int d\theta_j^a \exp\left\{i \int dx\, \Lambda(Q_j^a,\ G_{0j}^a,\ G_{jk}^a)\right\} \tag{37}$$

where

$$\Lambda(Q_i^a, G_{0i}^a, G_{ik}^a) = L(Q_i^a + G_{0i}^a, G_{ik}^a) - L(G_{0i}^a, G_{ik}^a) - \frac{\partial L(Q_i^a + G_{0i}^a, G_{ik}^a)}{\partial Q_j^b} Q_j^b \tag{38}$$

Expanding Λ in powers of Q_i^a we get

$$\Lambda(Q_i^a, G_{0i}^a, G_{ik}^a) = Q_i^a \Lambda_{ij}^{ab}(G_{0i}^a, G_{ik}^a) Q_j^b + O(Q^3) \tag{39}$$

where Λ_{ij}^{ab} is a local function (not a differential operator), and consequently the integral over Q_j is equal to unity within the dimensional regularization scheme, just as in the case discussed in Sec.2.

Thus for Z_H we obtain the expression

$$Z_H = \int dA_\mu^a (\partial_i A_i^a) \det(\partial_i \nabla_i^{ab}) \exp\left\{iL(G_{\mu\nu}^2)\right\} \tag{40}$$

which is exactly the generating functional Z_L in the Lagrangian formalism in the Coulomb gauge [2-4] .

This concludes the proof of the equivalence of Z_L and Z_H for the gauge theories $L = L(G_{\mu\nu}^2)$.

The demonstration of the equivalence of the Hamiltonian and Lagrangian formalisms in quantum field theory for $L(\phi)$ and $L(G_{\mu\nu}^2)$ encourages the hopes that they are equivalent in the general case.

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К ВОПРОСУ ОБ ЭКВИВАЛЕНТНОСТИ
ГАМИЛЬТОНОВА И ЛАГРАНЖЕВА ПОДХОДОВ В КВАНТОВОЙ
ТЕОРИИ ПОЛЯ

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