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LOOPS AND STRINGS IN QCD

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LOOPS AND STRINGS IN QCD

Concepts and methods of the dynamics of loops and strings in QCD are discussed and illustrated after the example of the model of random matrices interacting with quark fields. The first terms of $1/N$ expansion are constructed for this model.

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ПЕТЛИ И СТРУНЫ В КХД

Идеи и методы динамики петель и струн в КХД обсуждаются и иллюстрируются на примере модели случайных матриц, взаимодействующих с кварковыми полями. Построены первые члены $1/N$ разложения для этой модели.

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LOOPS AND STRINGS IN QCD

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Most of the physical laws are only approximate. The P and T symmetries are broken, the neutrino has mass, the proton is expected to decay etc. The new discoveries usually reveal tiny violations of apparantely absolute principles. The old theoretical schemes appear too narrow to fit the new facts and have to be generalized.

However it happened the other way around with the quark confinement. Here the Nature came out more perfect than we expected. There are still some sceptics, but for most of us the existing evidence for quark confinement is fairly convincing. This is the experimental evidence as well as results of numerical investigations of the lattice gauge theories. We are not going to discuss this evidence here, we simply take for grunted that quarks are permanently confined in QCD.

Whenever we find something perfect, which could have not been such according to our principles it is a good time to revise these principles. May be a narrower scheme can be found with no room for imperfections? For about 20 years fundamental physics was dealing with a wide scheme of general quantum field theory, which as we suspect now is never realized.

For some reason Nature exclusevely uses gauge theories.

Maybe there is something special in gauge theories which would enable us to go further than with general quantum field theory. Within standard perturbation theory this does not seem to be the case. On the contrary, the Faddeev-Popov-de Wit rules look like the general Feynman rules. Though very efficient in application to electroweak theories these rules hide all the geometric properties of the gauge fields, which make the gauge theories unique. Maybe there is another quantization procedure which would preserve and fully utilyse the geometric meaning of gauge theories.

All these aesthetic arguments would be immaterial were it not for the basic flaw of the standard quantization of gauge theories. It fails to describe the quark confinement. The Feynman graphs in QCD describe the quarks moving freely in space and exchanging gluons from time to time. The gluon exchange does not produce the constant attraction forces which as we beleive exist between quarks at large distances. Various mechanisms enhancing the Coulomb forces were proposed but non of them led to a quantitative theory. The best that can be done within perturbative quantization is to derive the sum rules for the Greens functions of various gauge invariant composite fields. With a bit of luck the sum rules work with accuracy 10-20% or even better. This, however, does not substitute the missing microscopic theory.

At this moment we have to say something about lattice gauge theories. Do they provide us with the theory of quark confinement? Lattice gauge theories are, indeed,capable of describing confinement, but they are too wide in other respects. The general lattice lagrangian contains infinitively many terms with corresponding constants in front. The continuum limit corresponds to all these constants simultaneously increasing at decreasing lattice spacing. As it was demonstrated by the recent Monte-Carlo simulations. This nit is not so smooth as one would like it to be. At large lattice spacing there are various spurious fluctuations-vortires, monopol etc. At small lattice spacing these fluctuations freese out. So... we are left with fluons as we want. There are some first-order phase transitions between the strong and weak coupling phases in parameter space. These phase transitions presumably preserve confinement, but we can no longer trust the strong coupling expansion- the only known analytic method in lattice gauge theories In particular the notion of the Wilson string connecting quarks in the strong coupling phase has to be revised in the relevant asymptotically free phase. The Monte-Carlo simulations are limited by computer time in the asymptotically free phase, when it comes to the most interesting infrared effects. The Lorentz symmetry cannot yet be reproduced by Monte-Carlo simulations.

So, neither perturbative QCD nor lattice gauge theories provide us with the analytic description of the infrared phenomena.

Do we really need such analytic theory? Would it be an aplied problem like the weather forecast the phenomenological models and numerical experiments would be sufficient.But in a fundamental problem like this we may expect some hidden mathematical beauty as well as some interesting physics. We do not only need the numbers, we also need the new language. After all QCD is not going to be the last confining gauge theory. Many great physical theories in the past arose as a result of attempts to achieve complete description of fundamental phenomena. So it seems worthwhile to look for a new quantization of gauge theories, which would make confinement natural.

From the pragmatic point of view one would like to have a perturbation theory with a small parameter and the zeroth approximation close to reality. One of the basic experimental facts in hadronic physics is absence of extra quark pairs in the low lying resonances. It is summarised phenomenologically by the Zweig rules. The meson consist of one quark pair and the barvon consist of three quarks. The laws of quantum field theory allow for arbitrary admixture of qq pairs since the number of particles is not conserved. The narrowness of resonances is a related phenomenon. Why does it take so long to create additional quark pair in the resonance decay? The universal explanation of these low energy phenomena (at least for mesons) is provided by the 1/N expansion first proposed by t'Hooft. The simple estimates abstracted from perturbative QCD yield N^{-1} for the widths of resonances and $N^{1-k/2}$ for the K -point meson amplitudes in QCD with N colors. The extra quark pairs in hadrons are damped by a factors $N^{-1/2}$ in the wave function.

In the real world N^{-1} is only 0.33. Is it small enough to apply the 1/N expansion? One can never tell in advance. Recall that the electron charge $e = \sqrt{4\pi/137} = 0{,}30$ but still the perturbation theory in QED works remarkably well. This is, of course, due to the factors $1/8\pi^2$ which appear in front of e^2 in perturbation theory (no such factors are present in the Lag -

rangian). The experimental validity of Zweig rules as well as theoretical calculations scattered in the literature lead to the estimate $(\pi N)^{-1} \sim 0.1$ of an effective expansion parameter. In absence of quarks only even powers would enter, so this would be almost as good as in QED. In the presence of quarks the odd powers will make it worse, but still reasonable.

We are discussing the expansion as if it is already constructed. In fact this turned out to be an outstanding theoretical problem, calling for new mathematical methods. The few steps made recently in this direction will be summarized in our review article.

The crucial step is the following one. The QCD can be quantized in such a way that 1/n expansion becomes the WKB expansion around some nontrivial "classical" solution", though the original functional integral of QCD was not dominated by any classical gauge field. Incidentaly this quantization is of the kind discussed in the beginning. It is peculiar for gauge theories with their geometry. For the general Q.F.T. such quantization does not exist, so the 1/N expansion in general Q.F.T. will be quite different. We believe that this quantization which we call loop dynamics is interesting by itself even desregarding applications to the 1/N expansion. Some new understanding of the gauge theories, in spirit of the Feynman ideas of space-time description of QED emerges within the framework of loop dynamics. The euclidean space time is implied throughout this paper. The basic dynamical object is an amplitude for the test particle to propagate along the given closed world line in the vacuum of a pure gauge theory. This propagation is influenced by fluctuating gluon field and an average amplitude $W(C)$ depending on the form of the loop is introduced. This is the well known Wilson loop. It turns out, that the closed functional equations for the set of multiple Wilson loops can be derived from original Shwinger Dyson equations of QCD. In general this is a kind of second quantized theory - the loops can be created from vacuum and multiple loops correlate. However at $N \to \infty$ the correlation decreases as N^{-2}, so the amplitude for several loops factorized to a product of individual Wilson loops. Hence the loop field with expectation value given by the Wilson loop becomes classical at $N = \infty$. It satisfies a certain nonlinear equation corresponding to equation for the set of planar graphs in old language. The planar graphs can, indeed, be found by direct iterations of this equation in the coupling constant.

What is much more important, the area law for the Wilson loop serves as a selfconsistent solution of this equation at large loops. The mechanism leading to the area law differs significantly from the Wilson mechanism of formation of strings in the high temperature expansion in the lattice gauge theory. In particular our mechanism is manifestly Lorentz invariant and it is compatible with asymptotic freedom. In fact the generalized planar graphs reproducing asymptotic freedom at small distances and the area law at large distances can be obtained by iterations of this equation.

Closer examination of the planar equation shows some notable deviations from the naive string model. The above mentioned modified planar graphs correspond to some nonlocal correlations at the world sheet of string. Is it possible to reproduce these nonlocal effects by adding some local internal degrees of freedom. This problem closely resembles the 2-dimensional Ising Onzager problem. The configurations of planar graphs at the world sheet of string can be viewd as borders between two phases coexisting at this surface. Developing this analogy the author found a peculiar system with elementary fermions elves at the world sheet of string. An analysis of the planar loop equation shows that this system solves planar QCD in the same sence as the free fermion system solved the Ising model. This elf theory has to be thoroughly investigated before any predictions for hadron spectra can follow. Some preliminary results will be reported here.

The purpose of this article is to provide an introduction to loop and string dynamics; some kind of a textbook. In original papers we usually omit the details and do not explain successive steps, leading to this or that result. Here, on the contrary, we concentrate on the basic notions and discuss alternatives. As for the applications and for the parallel developments we do not at all pretend to give a comprohensive review. We rather want to give the reader the method which can be applied to various problems. Still we mention some models and some applications which we find useful from this pedagogical point of view. Many of the results described in this article are original, they did not exist in the literature, at least in this form. However the ancther was influenced by some other works. Such papers are listed in the references.

We imagined three types of readers, when we were writing this paper. The mathematical physicist will find in the text as well, as in Appendices various conjectures which may be interesting to prove or improve.For us it seems important to find the mathematical foundation of the loop calculus developed here The pragmatic physicists are invited to apply the loop dynamics to the quark confinement problem. The potential abilities of the loop dynamics are by no means reached in the few applications described here. We believe that a quantitative theory can be worked out.

Hopefully there will be also a third category of readers-people like the postgraduate students who have more time and less prejiduce and may learn a new language simpli from curiousity. The whole text is addressed to the third category of readers, but in order to make it useful for the first two, some paragraphs are labeled by marks m or ρ . These are paragraphs which in authors opinion may be interesting for the correspon ding category of readers.

I. Loops and Strings in the Random Matrix Models (M,P)

In this Section we are going to introduce the basic ideas of loop dynamics by considering the 1/N expansion in a simple model, where the space consists only of one point. The model is nonrivial because the internal symmetry space will be taken to the same as in QCD.

The dynamics in flavor space in this model is a simplified but recognisable caricature of the dynamics in flacor x coordinate space in the real world as we view it within 1/N expansion.

Description of the Model

This is a generalization of the random matrix model solved by Bresin, Itzykson, Parizi and Zuber. There is an antihermitean matrix "gluon field" A^i_j and an anticommuting "quark" field q^{-ia}, q_{ia} where $i = 1,\dots,N$ is the color index, and $a = 1,\dots, N_f$ is the flavor index. To simplify equations we do not eliminate the trace of gluon field, so that our "gauge" group will be U(N) rather than SU(N). With one single points in space there is no difference between global and local transformations, so by gauge transformations we mean the U(N) rotation

$$A \to \Omega A \Omega^{-1} \tag{1.1}$$

$$q \to \Omega q \tag{1.2}$$

$$\bar{q} \to \bar{q}\,\Omega^{-1}. \tag{1.3}$$

The bilinear quark term in the Action may have the form:

$$\mathcal{L}_q = \bar{q}\,(A + B)\,q, \qquad (1.4)$$

where B is some matrix in flavor space. This will be the external spectator field, the source for gauge invariant Green functions in the model. At the same time this B may serve as a mass term.

The gluon part may contain an arbitrary potential:

$$\mathcal{L}_g = -\frac{1}{2g^2}\,\mathrm{tr}\;V\,(A). \qquad (1.5)$$

For definitness we may keep in mind the following simple choice:

$$V = (iA + A^2)^2, \qquad (1.6)$$

which leads to cubic and quartic vertices as in ordinary gauge theory. The term A^2 imitates the commutator term in the field strength (remember, that A is antihermitean)

Our problem in this model reduces to the calculation of the vacuum amplitude in the presence of aspectator field

$$Z\,(W) = \int dA\,d\bar{q}\,dq\,\exp\,(\mathcal{L}_g + \mathcal{L}_q). \qquad (1.7)$$

Whenever the model is soluble it can be solved by variety of methods. The original random matrix model (without quarks) was solved at $N=\infty$ by going to the gauge where A_i^j was diagonal and using the saddle point methods. There are only N variables A_i^i in this gauge.

The Faddeev-Popov determinant reduces to a Vandermonde determinant

$$\Delta = \prod_{i<j}\,(A_i^i - A_j^j)^2. \qquad (1.8)$$

The effective potential

$$V_{eff} = \frac{1}{2g^2} \sum_{i=1}^{N} V(A_i^i) - 2\sum_{i<j} \ln |A_i^i - A_j^j| \qquad (1.9)$$

posesses a nontrivi.. minimum (the master field). The details of this beautiful s..ution can be found elsewhere.

Here we apply the method of the loop equations which turnes out to be more efficient. The leading term as well as 1/N corrections will be found explicitely (some mathematical details are described in Appendix A). We are not interested in the details of the model, but rather the general picture of dynamics in flavor space. The ouput will be an analogy with the string model.

Loop Source

We are going to apply to this model the same sequence of transformations, which leads to the equations in QCD. Here we arrive at some functional equations of the same general structure but, of course, simpler. This will help to understand the strategy of the loop dynamics.

The first step is to eliminate quarks by Gaussian integration.

$$\int dq \exp(\mathcal{L}_q) = \mathcal{D}et(A+B) = \exp\left(\mathrm{Tr} \ln(A+B)\right) \qquad (1.10)$$

The determinant and trace here correspond to the complete flavor x color space. We have to separate color matrices from flavor matrices before going further. This can be achieved by the proper time representation

$$\ln(A+B) = \frac{1}{\varepsilon} + \gamma - \int_0^\infty \frac{dT}{T} T^{\varepsilon} e^{+AT} e^{+BT} \qquad (1.11)$$

The ultraviolet cutoff $\varepsilon \to 0$ introduces no problems, since there will be no divergences. The Euler constant γ will contribute to the vacuum energy would we wish to calculate it, but we don't. What is important for us at the moment is the factorization of the integrand on the right of (1.11).

The complete trace of (1.11) will reduce inside the integral to the product of the color trace which we denote as tr, and the flavor trace which we denote as Sp

$$\mathrm{Tr}\, \ln (A+B) = \mathrm{const} + \int_0^\infty J(T)\psi(T)\, dT \tag{1.12}$$

$$\psi(T) = \mathrm{tr}\, e^{AT} \tag{1.13}$$

$$J(T) = -T^{\varepsilon-1}\, \mathrm{Sp}\, e^{BT} \tag{1.14}$$

This factorization has the counterpart in the real world. The amplitude for the vacuum quark loop of the given form c will factorized as the Dirac amplitude times flavor amplitude times color amplitude. The last factor is the trace of the ordered exponential of the integral of gluon field along the loop, in short, it is the loop field.

In our model the quark never leaves the single point in space, but the proper time T varies. So loop is represented by an interval of proper time. The loop field reduces to (1.13).

Now we come to the most important point in loop dynamics. We observe that the quark contribution (1.12) to the effective action can be regarded as the source term for the loop field Quarks produce some particular loop sourse $J(T)$ implicitely depending on the spectator field.

The spectator field is a matrix, so this source depends on N_f parameters, but it is convenien* to generalize the model by considering arbitrary functions $\mathcal{J}(T)$. In the real world this will be the functional source, depending on the form of the quark loop. This generalization is necessary to obtain closed equation of motion.

The Loop Equation

Equations for the vacuum functional z $[\mathcal{J}(T)]$ follow from the the identity

$$0=\int dA \frac{\partial}{\partial A_i^j} (e^{At})_i^j \exp\left(\mathcal{L}_g + \int_0^\infty dT\, \mathcal{J}(T)\psi(T)\right). \qquad (1.15)$$

(The integral of the total derivative of a function, which vanishes at the (infinite) end points). We only have to calculate derivatives and reduce the terms. The derivatives are calculated as follows

$$\frac{\partial}{\partial A_i^j}(e^{At})_k^\ell = \int_0^t d\tau\, (e^{A\tau})_k^i\, (e^{A(t-\tau)})_j^\ell \qquad (1.16)$$

$$\frac{\partial}{\partial A_i^j} \operatorname{tr} f(A) = (f'(A))_j^i . \qquad (1.17)$$

Reducing the terms we find in front of exponential

$$\int_0^t d\tau\, \psi(\tau)\psi(t-\tau) - \frac{1}{2g^2}\operatorname{tr}\left(V'(A)e^{At}\right) + \qquad (1.18)$$

$$+ \int_0^\infty dT\, T\, \mathcal{J}(T)\, \psi(T+t).$$

This can further be reduced by introducing the differential operator:

$$L\psi(t) = \frac{1}{2} V'\left(\frac{\partial}{\partial}\right)\psi(t) = \frac{1}{2}\operatorname{tr}\left(V'(A)e^{At}\right) \qquad (1.19)$$

For the quartic potential V this L will be the differential operator of the third degree with coustant coefficients. In the real world (see later) this will be some peculiar combination of third functional derivatives.

The last step in derivation of the loop equation is the standard replacement $\psi \to \delta/\delta\mathcal{J}$ namely:

$$\psi(t)\exp\left(\int_0^\infty dT\,\mathcal{J}(T)\,\psi(T)\right) = \tag{1.20}$$
$$= \frac{\delta}{\delta\mathcal{J}(t)}\exp\left(\int_0^\infty dT\,\mathcal{J}(T)\,\psi(T)\right).$$

(In the real world this step will be quite unusual, since $\mathcal{J}$ itself will be a functional!) Collecting all the terms we arrive at the following loop equation of motion

$$g^{-2} L\,\frac{\delta Z}{\delta\mathcal{J}(t)} = \int_0^t d\tau\,\frac{\delta^2 Z}{\delta\mathcal{J}(\tau)\,\delta\mathcal{J}(t-\tau)} + \tag{1.21}$$
$$+\int_0^\infty dT\,T\,\mathcal{J}(T)\,\frac{\delta Z}{\delta\mathcal{J}(T+t)}\,.$$

This equation does not contain the number N of colors. It enters through the initial condition

$$\frac{\delta Z[\mathcal{J}]}{\delta\mathcal{J}(0)} = N Z[\mathcal{J}], \tag{1.22}$$

which corresponds to

$$\psi(0) = \operatorname{tr} 1 = N. \tag{1.23}$$

Now we see, why the functional source was necessary. It allowed for analytic continuation to arbitrary N, necessary for $1/N$ expansion. Or to put it in another words, the matrix A of an infinite rank is equivalent to a function, so the source have to be the function as well. In the continuous space the matrix gauge field of an infinite rank N will be equivalent to the loop functional and will require a loop functional source. We came to the starting point for the 1/N expansion.

1/N Expansion

The number N of colors which enters through the initial condition (1.23), can be put to denominator by the following simple transformations. Redefine the coupling constant

$$\lambda = Ng^2 = indep\ (N) \tag{1.24}$$

and introduce the normalized loop field

$$\phi(t,[J]) = N^{-1} Z^{-1} \frac{\delta Z}{\delta J(t)} \tag{1.25}$$

$$\phi(0,[J]) = 1. \tag{1.26}$$

This is N^{-1} times expectation value of previously defined loop field in presence of the loop source. The second order linear equation (1.21) for the vacuum functional reduces to the first order nonlinear equation for this loop field in the same way as the Shrodinger equation reduces to the first order nonlinear equation for the logarithmic derivative of the wave function. In the latter case we obtain the WKB expansion by iterating in Planks constant in front of derivative. Here the same method will result in 1/N expansion. Equation for loop field reads

$$\lambda^{-1} L\phi(t) = \int_0^t d\tau\, \phi(\tau)\phi(t-\tau) + N^{-1}\Big(\int_0^\infty dT\, T \times \\ \times J(T)\phi(T+t) + \int_0^t d\tau \frac{\delta\phi(t-\tau)}{\delta J(\tau)}\Big). \tag{1.27}$$

For brevity we omitted the functional argument of $\phi(t,[J])$

This is the form of the loop equation we were looking for. The field theoretical analogue would be the ϕ^3 theory in the loop space. However this analogy is incomplete and it is important to realize it. There is no simple functional integral over fields $\phi(t)$ corresponding to this functional equation. The difference between this equation and Shwinger equation for the field theories with independent fields $\phi(t)$ reduces to the dependence of the source term $\int d\Pi J\phi$ on the field $\phi(t)$. This is so because the field is not independent, and we derived these equations from original Shwinger equations for the A -field. In the real world it would be even worse than that, since there will be certain relations between loop fields for intersecting loops. Still the closed equations for the loop fields exist in continuum space as well as in this o-dimensional model.

What is really surprizing in both cases is the possibility of complete elimination of coloured objects from the theory. Note that we are discussing the case of finite N. So the loop equation provides an alternative language which describes the same system in completely different physical terms. Such phenomena occur in 2-dimensional soluble models. Recall exact equivalence between Sine-Gordon and Thirring models. There also two apparantely different languages described the same physical system. We shall return to these questions later on. Let us continue with the 1/N expansion. It is now quite straightforward.

In the zeroth order we have to solve nonlinear equation

$$\lambda^{-1} L\, \phi_0(t) = \int_0^t d\tau\, \phi_0(\tau)\, \phi_0(t-\tau) \qquad (1.28)$$

with appropriate boundary conditions and infinity. Among the va·iety of solutions we have to choose the one which corresponds to the free quark theory at vanishity coupling, i.e.

$$\Phi(t,[J])\Big|_{\lambda=0} = 1. \tag{1.29}$$

This is an asymptotic freedom condition which should also be satisfied in continuum theory. This solution is found in Appendix A by means of Laplas transformation which diagonalizes our equation.

In the first order $\Phi = \Phi_0 + N^{-1}\Phi_1$ one arrives at the linearized equation:

$$\lambda^{-1} L\Phi_1(t) = 2\int_0^t d\tau\, \Phi_0(t-\tau)\Phi_1(\tau) + \int_0^\infty dT\, T J(t)\Phi_0(T+t). \tag{1.30}$$

Solution is found in Appendix A.

In higher order we have to solve the same linearized equation with different inhomogeneous terms. All we need in order to find explicitely these higher terms is the Green's function of linearized equation, which is also found in Appendix A. So there is an algorithm for the 1/N expansion. There is an interesting physical picture behind this algorithm.

Analogy with String Theory

This picture is well known within the perturbation theory. When the Feynman graphs are classified by powers of N^{-1} at fixed λ, they arrange to the sets of given topology, i.e. the sets of graphs, which can be drawen without intersections at surfaces with given topology. Leading term corresponds to the surface without handles and with the minimal number of quark loops for given type of sources under considerations. In higher orders in $1/N$ each handle yields N^{-2} and each vacuum quark loop yields N^{-1} in connected amplitude. These statements have to be ex - plained if the reader is not familiar with them. Instead of repeating the standard formal arguments we may do something better in our model, namely we shall give some physical meaning to these mysterious surfaces. These will be genuine surfaces in flavor space and the successive terms of $1/N$ expansion will involve the surfaces with given topology. Remeber, that quark has flavor index, so the evolution of quark state can be regarded as propagation in flavor space.

To be specific, consider the qq expectation value. According to the chain rule.

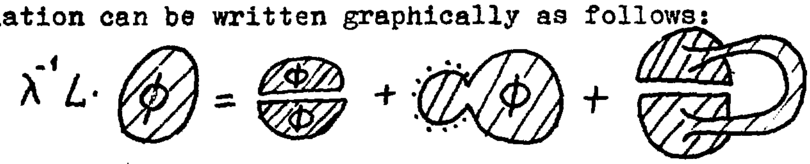
$$\frac{1}{N}\langle \bar{q}^a q_a \rangle = \frac{1}{N}\frac{\partial \ln Z}{\partial B_a^a} = \frac{1}{N_0}\int_0^\infty dT \frac{\delta \ln Z}{\delta J(T)} \frac{\partial J(T)}{\partial B_a^a} =$$

$$= -\int_0^\infty dT \phi(T)\, Sp\left(e^{BT}\right). \qquad (1.31)$$

By representing

$$Sp\, e^{BT} = \left(e^{\varepsilon B}\right)_a^{a_1}\left(e^{\varepsilon B}\right)_{a_1}^{a_2} \cdots \left(e^{\varepsilon B}\right)_{a_L}^{a} \qquad (1.32)$$

with infinitesimal $\varepsilon = T/L$ we obtain the sequence of points $a, a_1, \ldots a_L, a$ in the flavor space. These points describe the closed loop, since the initial and final points coincide. The summation over intemediate indices corresponds to sum over all closed loop in flavor space. The free quark will have the propagator (1.32), but quark in the gluon vacuum gets an additional factor $\phi(T)$ At finite N this factor will depend on the loop source, I, which means that extra vacuum pairs influence the quark under consideratiòn. At infinite N this influence disappears. The dependence of ϕ on J starts from the terms $\sim N^{-1}$ which were considered above. One may associate with the leading term ϕ° the simple surface(disc) the flavor space, bounded by the quark loop. This is a world sheet of string with the given world line of the ends. The quark pair was born at the point a, then propagated along the given loop, and the string propagated along the corresponding sheet (Fig.1). So far this interpretation is a matter of definition. However, in the next orders we shall observe, that interpretation is selfconsistent. The surfaces with holes and handles will appear and fit the same string picture. This can be seen from the loop equation (1.27). This equation can be written graphically as follows:

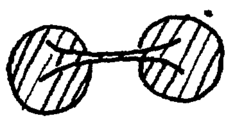


(1.33)

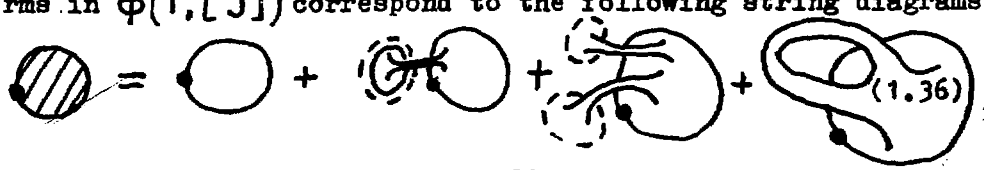
Here the proper time t is associated with area the corresponding window. The areas $t-\tau$, τ in the first term add up to t, areas T and t enter the second term. The thin handle in the third term has no area. The dotted line in the second term correspond to the quark propagator

 $= N^{-1} T \mathcal{J}(T) = -N^{-1}\int \rho e^{\beta T}$ (1.34)

and the cylindric surface in the last term corresponds to the correlation function

 $= N^{-1}\dfrac{\delta\,\phi\,(t_1)}{\delta\,\mathcal{J}\,(t_2)} \equiv N^{-2} K(t,t_2).$ (1.35)

The last two terms in the loop equation (1.33) generate respectively the holes and handles in the surfaces. The N^{-1} and N^{-2} terms in $\phi(T,[\mathcal{J}])$ correspond to the following string diagrams



(1.36)

Here the white surfaces correspond to $\phi_0(t)$, $K_0(t_1 t_2)$ etc. The last term with the handle was generated by the last term in (1.33), and the terms with quark loops were generated by the second term in (1.33). The reader may work out the details himself using the formulas of Appendix A. For our purpo ses it is important to know in principle that the terms of 1/N expansion can be associated with the string propagation. In the 0 - dimensional modelthere is only a flavor space to propagate in, but it is natural to expect that in the continous space the material string propagate. These arguments, however, tell us nothing about string action. It need not coincide with the area of the world surface.

Conclusions

i) The systematic 1/N expansion in generalized random matrix model can be constructed for arbitrary coupling constant by means of the loop equations

ii) The successive terms can be interpreted in terms of the string propagation. The string with quark at the ends moves in flavor space in this model.

In the real world it is expected to propagate in complete physical space flavor x coordinates x spin.

iii) The O-dimensional model is useful to get some insight for the nonperturbative phenomena in continous theory.

Appendix A

Solution of 0-dimensional Loop Equations

Consider an integrodifferential equation

$$\hat{L}\left(-i\frac{\partial}{\partial t}\right)\phi(t)+\lambda\int^{t} d\tau\,\phi(\tau)\,\phi(t-\tau) \qquad \text{(A.1)}$$

$$0\le t<\infty$$

with the initial condition

$$\phi(0)=1 \qquad \text{(A.2)}$$

Only solutiors bounded at infinity have the physical meaning, since there is an iequality

$$|\phi(t)|\le 1, \qquad \text{(A.3)}$$

which follows from positivity of the measure and antihermiticity of the A-matrix in functional integral (1.1)

In our particular model L is a cubic polynomial

$$L_3(\omega)=\omega\,(1+\omega)(1+2\omega), \qquad \text{(A.4)}$$

but we shall construct here the general solution for

$$L_n(\omega)=\left(\ell_1\omega+\ell_2\omega^2+\ldots+\ell_n\omega^n\right). \qquad \text{(A.5)}$$

Naturally, only odd n and positive ℓ_n make sence in original problem, where $L(A)$ is the derivative V' of the potent[illegible] $V(A)$

The initial condition (A.2) fies only one parameter of required Cauchy data, and the Taylor series

$$\phi(t,A)=\sum_{n=0}^{\infty}\phi_K(\lambda)\frac{(it)^K}{K!}\ ,\quad \phi_0=1 \qquad \text{(A.6)}$$

contains (n-1) unknown coefficients

$$\phi_1(\lambda),\ \phi_2(\lambda),\ \cdots,\ \phi_{n-1}(\lambda). \qquad \text{(A.7)}$$

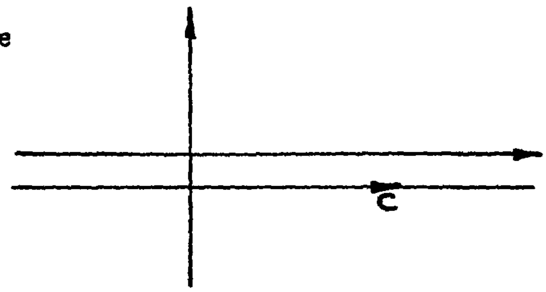
Given these coefficients, the remaining coeffi[illegible] can be found from recurrence equation[illegible]

$$\sum_{K=1}^{n}\ell_K\,\phi_{K+\tau}=\lambda\sum_{s=1}^{\tau-1}\phi_{s-1}\,\phi_{\tau-s} \qquad \tau=1,2,\ldots$$

In order to find an exact solution let us perform the Laplace transformation

$$\theta(t)\phi(t) = \int_C \frac{d\omega}{2\pi i} e^{i\omega t} F(\omega), \qquad (A.9)$$

where contour C goes parallel the real axes in the lower semiplane



The inverce transformation reads

$$F(\omega) = i\int_0^\infty dt\, e^{-i\omega t} \phi(t). \qquad (A.10)$$

Due to our boundary condition for $\phi(t)$ $F(\omega)$ is holomorphic in lower semiplane.

The Taylor series (A.6) corresponds to Laurant series in Laplace transform

$$F(\omega) = \sum_{K=0}^{\infty} \omega^{-K-1} \phi_K. \qquad (A.11)$$

Let us integrate (A.1) with $e^{-i\omega t}$ for $t=0$ to $t=+\infty$

$$\lambda F^2(\omega) = i\int_0^\infty dt\, e^{-i\omega t} L\left(-i\frac{\partial}{\partial t}\right)\phi(t). \qquad (A.12)$$

Now we integrate by parts and arrive at the following equation

$$\lambda F^2(\omega) = L(\omega)F(\omega) - Q(\omega), \qquad (A.13)$$

where Q is some polynomial of degree n-1

$$Q = q_0 + q_1\omega + \dots + q_{n-1}\omega^{n-1}. \qquad (A.14)$$

The higher term is known

$$q_{n-1} = \ell_n \tag{A.15}$$

and the remaining terms are related to unknown coefficients (A.7).

$$q_s = \sum_{m=1}^{n-s} \phi_{m-1} \, \ell_{s+m}. \tag{A.16}$$

The proper solution of quadratic equation (A.13) reads

$$F(\omega) = \frac{L(\omega)}{2\lambda} - \sqrt{\left(\frac{L(\omega)}{2\lambda}\right)^2 - \frac{Q(\omega)}{\lambda}}\,. \tag{A.17}$$

The negative sign follows from the initial condition, i.e. from the Laurant series at $\omega \to \infty$ (A.11)

Rewriting (A.17) as

$$F(\omega) = \frac{2Q}{L + \sqrt{L^2 - 4\lambda Q}}\,, \tag{A.18}$$

we may easily check that the Laurant series has a correct structure.

Now we should determine the unknown coefficients from some physical requirement.

There may be several solutions for coefficients depending of the value of the coupling constant we are interested in particular solution which can be expanded in perturbation theory in λ (asymptoticaly free solution)

This solution can be constructed as follows.

Choose the coefficients of Q in such a way, that 2n-2 among 2n roots of expression inside the square root arrange in coinsiding pairs, i.e.

$$L^2 - 4\lambda Q = M^2(\omega - a)(\omega - b), \tag{A.19}$$

where M is the polynomial of degree n-1

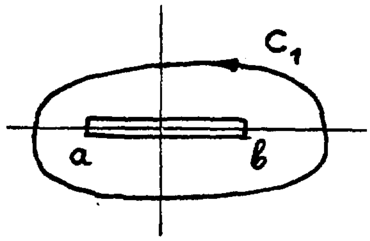
$$M = m_0 + m_1\omega + \ldots + m_{n-1}\omega^{n-1}. \tag{A.20}$$

One may compare the known coefficients in $\omega^{2n} \ldots \omega^{n-1}$ in l.h.s of (A.19) with corresponding coefficients in r.h.s. and get a system of n+2 equations for n+2 coefficients m_K, a, b. The coupling constant would enter via term $-4\lambda \ell_n \omega^{n-1}$ in the l.h.s. After solving this system the coefficients of Q are fixed by comparison of lower powers of ω.

Fortunately, there exists an explicit solution for this M. Namely

$$M(\omega) = +\frac{1}{2\lambda} \oint_{C_1} \frac{d\omega}{2\pi i} \frac{L(\omega) - L(\omega')}{(\omega-\omega')\sqrt{(\omega'-a)(\omega'-b)}}, \tag{A.21}$$

where the contour C_1 encircling the interval (a,b) in the complex ω - plane.



(A.22)

By constructing this is polynomial in ω of degree n-1, since $L(\omega') - L(\omega)$ can be divided by $\omega' - \omega$.

$$\frac{L(\omega) - L(\omega')}{\omega - \omega'} = \sum_{n=0}^{n} \ell_n \sum_{\tau=0}^{K-1} \omega'^{K-\tau-1} \omega^{\tau} = \sum_{\tau=0}^{n-1} \omega^{\tau} \sum_{u=\tau+1}^{n} \ell_K (\omega')^{u-\tau-1}. \tag{A.23}$$

The function $F(\omega)$ with this choice of M in (A.17),(A.19) reads

$$F(\omega) = \frac{L(\omega)}{2\lambda} - \frac{M}{2\lambda}\sqrt{(\omega-a)(\omega-b)} =$$

$$= (2\lambda)^{-1}\left\{L(\omega) + \sqrt{(\omega-a)(\omega-b)} \oint_{C_1} \frac{d\omega'}{2\pi i} \frac{L(\omega')-L(\omega)}{(\omega-\omega')\sqrt{(\omega'-a)(\omega'-b)}}\right\} \tag{A.24}$$

$$= (2\lambda)^{-1}\sqrt{(\omega-a)(\omega-b)} \oint_{C_1} \frac{d\omega'}{2\pi i} \frac{L(\omega')}{(\omega-\omega')\sqrt{(\omega'-a)(\omega'-b)}}.$$

The last line followed from the residue at $\omega' = \omega$ in the term with $L(\omega)$ in the integral.

This function $F(\omega)$ can be expanded in Laurant series at

$$F(\omega) = F_0 + F_1\omega^{-1} + F_2\omega^{-2} + \dots \tag{A.25}$$

The coefficient F_0 should vanish, since $\phi(t=0)$ is finite. This yields an equation between a and b

$$0 = \oint_{c_1} \frac{d\omega'}{2\pi i} \frac{L(\omega')}{\sqrt{(\omega'-a)(\omega'-b)}} = \tag{A.26}$$

$$= \sum_{0 \le K+i \le n} \binom{-1/2}{K}\binom{-1/2}{i}(-a)^K(-b)^i \, \ell_{K+i} .$$

The coefficient F_1 should be equal to 1 according to our initial condition. This relates a and b to Z

$$2\lambda = \oint_{c_1} \frac{d\omega'}{2\pi i} \frac{L(\omega')\,\omega'}{\sqrt{(\omega'-a)(\omega'-b)}} = \tag{A.27}$$

$$= \sum_{1 \le K+i \le n+1} \binom{-1/2}{K}\binom{-1/2}{i}(-a)^K(-b)^i \, \ell_{K+i-1} .$$

There may be several branches of solutions for $a(\lambda)$, $b(\lambda)$, but there is a unique perturbative branch which stands as follows

$$(a, b) = \pm\sqrt{\frac{\lambda}{\ell_1}} - \frac{2\lambda \ell_2}{\ell_1^2} + \dots \tag{A.28}$$

$$F(\omega) \to \omega^{-1} + \lambda/(\ell_1 \omega^2) + \dots \tag{A.29}$$

One may check this by direct substitution of (A.28),(A.29) into (A.26), (A.27)

So, the integrodifferential equation is reduced to two algebraic equations for the coefficients a,b.

One could take the following Ansatz for the solution of (A.1)

$$\phi(t) = (2\lambda)^{-1} \int_c \frac{d\omega}{2\pi i} \int_{c_1} \frac{d\omega'}{2\pi i} \frac{L(\omega')e^{i\omega t}}{(\omega-\omega')} \frac{\omega'}{\omega} \sqrt{\frac{(\omega-a)(\omega-b)}{(\omega'-a)(\omega'-b)}} \tag{A.30}$$

This Ansatz appears to satisfy the equation (A.1) in virtue of the condition (A.26). The second condition (A.27) follows from the normalization at t=0

As for the behaviour at $t \to \infty$ it is correct for small enough $\lambda > 0$ and $\ell_1 > 0$ when a and b are real.

Finally consider the inhomogeneous equation

$$iL\left(-i\frac{\partial}{\partial t}\right)Y_1(t) + 2\lambda\int_0^t d\tau\ \phi_0(\tau)\ Y_1(t-\tau) = G(t). \tag{A.31}$$

Proceedings as before we find for the corresponding Laplace transform

$$\tilde{Y}(\omega) = \frac{\tilde{G}(\omega) - R(\omega)}{M(\omega)\sqrt{(\omega-a)(\omega-b)}}. \tag{A.32}$$

Here $R(\omega)$ is the polynomial of degree n-1

$$R = \tau_0 + \tau_1\,\omega + \dots + \tau_{n-1}\,\omega^{n-1}. \tag{A.33}$$

In our case we should satisfy the initial condition

$$\lim_{\omega\to\infty}(\omega\,\tilde{Y}(\omega)) = Y(0) = 0, \tag{A.34}$$

which yields

$$\tau_{n-1} = 0 \tag{A.35}$$

The remaining (n-1) coefficients $\tau_0, \dots \tau_{n-2}$ should be chosen to cancel the poles at ω_i where

$$M(\omega_i) = 0. \tag{A.36}$$

The unique choice is given by

$$R(\omega) = \int_{C_2}\frac{d\omega'}{2\pi i}\ \frac{\tilde{G}(\omega')}{(\omega-\omega')}\ \frac{M(\omega)}{M(\omega')}, \tag{A.37}$$

where contour C_2 encircles anticlockwise all the roots of $M(\omega)$ but the singularities of $G(\omega')$. Calculating the integral by residues at these poles we find

$$R(\omega) = \sum_i \frac{\tilde{G}(\omega_i)}{M'(\omega_i)}\ \frac{M(\omega) - M(\omega_i)}{\omega - \omega_i}. \tag{A.38}$$

By construction this is the polynomial of degree n-2 taking the same values as G at ω_i as it should do.

So we find

$$\tilde{Y}(\omega) = \frac{1}{\sqrt{(\omega-a)(\omega-b)}}\left\{\frac{\tilde{G}(\omega)}{M(\omega)} + \int_{c_2} \frac{d\omega'}{2\pi i}\,\frac{\tilde{G}(\omega')}{(\omega'-\omega)M(\omega')}\right\} = \tag{A.39}$$

$$= \frac{1}{\sqrt{(\omega-a)(\omega-b)}} \oint_{c_3} \frac{d\omega'}{2\pi i}\,\frac{\tilde{G}(\omega')}{(\omega-\omega')M(\omega')},$$

which is an exact solution of the inhomogeneous with correct analytic properties (contour c_3 encircles the singularities of $\tilde{G}$)

In the loop space

$$Y(t) = \int_a^b \frac{d\omega}{\pi}\, e^{i\omega t} \frac{1}{\sqrt{(\omega-a)(\omega-b)}} \cdot \oint_{c_3} \frac{d\omega'}{2\pi i}\,\frac{\tilde{G}(\omega')}{(\omega-\omega')M(\omega')}. \tag{A.40}$$

Now we may rewrite the first $\frac{1}{N}$ correction to the loop field as follows

$$\phi(t) = \int_0^\infty dt'\, J(t')\, K(t,t'), \tag{A.41}$$

where the loop loop correlation function

$$K(t,t') = \langle \phi(t)\, \phi(t') \rangle \tag{A.42}$$

is given by the following integral

$$K(t,t') = t' \int_0^\infty d\tau\, \phi_0(t'+\tau) \int_{c_1} \frac{d\omega}{2\pi i}\,\frac{e^{i\omega t}}{\sqrt{(\omega-a)(\omega-b)}} \int_{c_3} \frac{d\nu}{2\pi}\,\frac{e^{-i\nu\tau}}{(\omega-\nu)M(\nu)}. \tag{A.43}$$

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