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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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A.A. MIGDAL

LOOP KINEMATICS

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КИНЕМАТИКА ПЕТЕЛЬ

Вводятся основные операторы, действующие в пространстве петель, обсуждается топология этого пространства и свойства петлевых функционалов стоксовского типа. Разрабатываемое здесь параметрически инвариантное петлевое исчисление используется в динамике петель.

Ереванский физический институт
Ереван 1982

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LOOP KINEMATICS

Basic operators acting in the loop space are introduced. The topology of this space and properties of the Stokes type loop functionals are discussed. The parametrically invariant loop calculus developed here is used in the loop dynamics.

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Yerevan 1982

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LOOP KINEMATICS

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II. Loop Kinematics. (m)

In this section we classify loop fields and study the related topology of loop space, introduce the basic operators which describe the motion in this space and investigate various properties of these operators.

All these kinematical relations will be utilized later in loop dynamics and in the string solutions of the loop equations The geometric language (loop calculus) not only simplifies notations, it helps to visualize the motion of loops. Without this loop dynamics would be as cumbersome as the old perturbation theory in Q.E.D.

Loop Space

A closed line in co-ordinate space can be described by a periodic function

$$C: \quad X_\mu = C_\mu(t) = C_\mu(t+T) \tag{2.1}$$

The parameter t is called the proper time. It can be chosen at will, say it may be identified with the length $\left(dt = \sqrt{dC_\mu^2}\right)$. For our purposes, however, it will be most convenient not to specify the proper time at all, since our functionals will be parametric invariant

$$C_\mu(t) \rightarrow C_\mu(F(t)) \tag{2.2}$$

$$F'(t) > 0 \tag{2.3}$$

The latter condition means that the overall order of the points orientation at the curve is preserved. It is important since noncommuting matrices will be ordered along the curve. So we may define the loop as a family of periodic functions which differ by reparametrizations preserving orientation. By definition each loop represents a point in loop space. Therefore, the loop space is a space of periodic functions factorized by the reparametrization subspace.Parametric invariance is of paramount importance for the loop dynamics.

Later on, in framework of string theory, this parametric invariance will be part of the general covariance of the string dynamics. Here we develop a special loop calculus which is manifestly parametric invariant. Whenever you insist on something you loose something else,in our case we do not care about smoothness of the loop. Only the continuity is important, since the original gauge invariance would break for the open loop. The heuristic reason for including irregular loops is given by the interpretation of loop as a quark world line. The dominant world lines are irregular in quantum mechanics. The loop may also interest itself. This turns out to be important,so we discuss the self-intersecting loops in some detail. Self-inter sections occur at all the points where

$$C_\mu(t_1) = C_\mu(t_2) \tag{2.4}$$

but

$$t_1 \not\equiv t_2 \pmod{T} \tag{2.5}$$

There may be multiple self -intersections corresponding to several coinciding co-ordinates $C(t_1) \ldots\ldots\ldots C(t_e)$.

Self-intersecting loops enter the equations of quantum theory, so we cannot neglect them.

In this section we discuss the kinematical aspects of self-intersections i.e. the topology of loop space.

Classification of Loop Fields

Let us classify the gauge invariants associated with various loops. The simplest of all is the famous loop field.

$$\Phi(c) = \frac{1}{N} \operatorname{tr} \ \mathcal{U}(C_{xx}) \tag{2.6}$$

$$\mathcal{U}(C_{xx}) = P\exp\left(\int_{C_{xx}} A_\mu dx^\mu\right) \tag{2.7}$$

As was already mentioned, the function $C_\mu(t)$ need not be differentiable, so $dx_\mu \neq C_\mu\, dt$. In this case the loop product can be defined as follows:

$$\mathcal{U}(C_{xx}) = \lim_{K\to\infty} (1 + A(x_1)dx_1)\ldots(1 + A(x_K)\,dx_K) \tag{2.8}$$

Here $x_K = x(t_K)$ are ordered points at the loop $x_1 = x_\mu = x$, and $dx_K = x_{K+1} - x_K$

An alternative definition reads

$$U(C_{00}) = \sum_{n=0}^{\infty} \int_{C_{00}} dx^{(1)}_{\mu_1} \int_{C_{10}} dx^{(2)}_{\mu_2} \cdots \int_{C_{K+1,0}} dx^{(K)}_{\mu_K} \times \tag{2.9}$$

$$\times A_{\mu_1}(x_1) \ldots\ldots A_{\mu_K}(x_K)$$

The path Cab is defined as part of C from a to b. There are points $X_1 \ldots\ldots X_K$ at Coo. One may consider these points as points at a unit circle in the complex plane.

$$z = exp\,(2\pi i\; t/T) \tag{2.10}$$

Due to periodicity $C_\mu(t)$ can be regarded as a single valued function of z. Note that the loop trace (2.6) does not depend on the choice of the origin $t=0$ of parametrization. This loop trace depends on the loop as a geometric object, rather than on the function $C_\mu(t)$. The geometric language will better fit the purposes of loop dynamics. From the geometric point of view the loop C is the mapping of the unit circle into the Euclidean Space.

Such mappings can be classified according to the winding number n. The loop may be transversed n times when the point goes along the unit circle. This can be described by introducing a complex variable z^n, or the proper time nt. We denote such loops by C^n.

$$C^n : \quad X_\mu = C_\mu(nt) \tag{2.11}$$

Negative n would correspond to a reoriented loop. In particular.

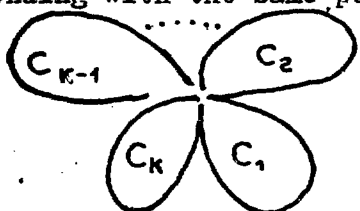
$$C^{-1} : \quad X_\mu = C_\mu(nt) \tag{2.12}$$

Consider the multiple loop traces. They can be expressed in terms of powers of the loop product.

$$\phi(C^n) = \frac{1}{N}\ \mathrm{tr}\ \mathcal{U}^n(C_{xx}) \tag{2.13}$$

From the geometric point of view this winding number may-be an arbitrary integer, but in the gauge theory there is only a finite number of independent fields $\phi(C^n)$, corresponding to the number of independent eigenvalues of $\mathcal{U}$. Namely, in $U(N)$ gauge theory there are N independent eigenvalues, all at the unit circle, and in $SU(N)$ theory there are $(N-1)$ independent eigenvalues,due to the unimodularity condition. By the way note that these eigenvalues by themselves map the loop onto the unit circle so altogether we have the mapping of the unit circle onto itself. This is so far as we disregard self-intersecting loops.

With self-intersections the set of invariants will be much richer. Consider the loop with petals all starting and ending with the same point.



In this case one may introduce winding numbers as follows

$$\begin{aligned}&\phi[C_1^{n_1} C_2^{n_2} \ldots\ldots C_K^{n_K} C_1^{m_1} C_2^{m_2} \ldots . C_K^{m_K} \ldots] \\ &= \frac{1}{N}\mathrm{tr}\,[\mathcal{U}^{n_1}(C_1)\,\mathcal{U}^{n_2}(C_2)\ldots\mathcal{U}^{n_K}(C_K)\ldots]\end{aligned} \tag{2.14}$$

The petal C_1, is transversed n times, then C_2 is transversed n_2 times and so on, but after the last petal is transversed n_K times one may start all over again. Therefore, winding numbers is not enough, and one has to introduce an infinite number of winding numbers.

This rich set of loop fields contains $\sim N^2$ degrees of freedom as the following simple counting shows. Let us use invariance of the trace and perform the unitary transformation which diagonalized, say, $\mathcal{U}(C_1)$ This unitary transformation contains N^2-N gauge parameters, which in addition to N eigenvalues describe the unitary matrix $\mathcal{U}(C_1)$ [in the SU(N) case there will be (N-1) eigenvalues and N^2-N gauge parameters adding up to (N^2-1) parameters $\mathcal{U}(C_1)$] The total number of parameters in all matrices $\mathcal{U}(C_1), \ldots \mathcal{U}(C_K)$ will be $K(N^2-1)$ in the SU(N) case). Substituting the number N^2-N of gauge parameters we find $(K-1)N^2+N$ independent traces in the U(N) case and $(K-1)N^2+N-K$ in the SU(N) case.

Explicit kinematical relations between the multiple loop traces were written down by Mandelstam. For the reader's convenience we reproduce these relations in appendix B. It is interesting that a product of $n(> N)$ loop traces in U(N) or SU(N) reduces to the superposition of lower products with n=1,2, ..., N in virtue of these relations.

An unpleasant property of this reduction is the presence of arbitrary "wires" connecting loops. The independence of the product of the L.H. side of the form of these wires implies some relations between the loop traces on the R.H. side.

We do not discuss these relations in the basic text, since no utilization was found so far.

As for the above counting the implications of it are purely negative, but important. We see that at $N\to\infty$ most of the degrees of freedom of the gauge field are hidden in the self - intersecting loop traces. Would we forget about them, we would deal with a theory with O(N) degrees of freedom.

In this case there would be no need for loop dynamics, since the direct saddle point method would be applicable. Later we shall see that most of the loop dynamics is related just with self-intersecting loops.

Stokes Type Functionals

The motion in ordinary space is described by a trajectory $X(t)$. This is a sequence of points labeled by a parameter t. The given loop c just describes motion of the test particle in co-ordinate space, as we shall discuss later on. Now we are going to talk about something else. A given loop represents a point in loop space, so we may introduce the motion in loop space as a variation of the form of the loop. In other words we go from trajectory to trajectory rather from point to point.

In principle this can be described by standard functional analysis, but we prefer to develop a special language (loop calculus) in our case. One reason why we do not use the functional derivatives $\delta \Phi / \delta c(t)$ is lack of smoothness of our loops. The cusps and self intersections will play an important role below. The ordinary functional derivatives do not exact in a mathematical sence. This would not stop the physicist were it not for the second reason. The second reason for abandoning the ordinary functional derivatives is the parametric invariance. There are some special properties of parametric invariant functionals, which would enable us to go much further than with the general functionals. It would be difficult to utilize these properties with the standard language. This is why we develop loop calculus.

To be specific, the loop functionals we deal with have the following form.

$$F(C) = \sum_{n=0}^{\infty} \int_{C_{00}} dx^{(1)}_{\mu_1} \int_{C_{10}} dx^{(2)}_{\mu_2} \cdots \int_{C_{n-1,0}} dx^{(n)}_{\mu_n} \times \tag{2.15}$$

$$\times F(\mu_1, x_1, \ldots \mu_n, x_n)$$

This is the same as for the loop field (2.6),(2.9), but with arbitrary coefficient functions F in place of the traces of the product of gauge fields. The origin Xo of parametrization at the loop is irrelevant since the coefficient functions do not depend on Xo. These functions are assumed to be cyclic symmetric, which is not a restriction due to the cyclic symmetry of the integral. Another property of these functions is implied in (2.15) comes out to be crucial. These functions depend on the points Xi in Euclidean space rather than on the trajectories C(t) in this space. In other words these are ordinary functions rather than functionals. In the integral they enter at the loop C, but there should be some simplifications due to the fact that they can be analytically continued from the loop to ordinary space. This subtle difference is difficult to utilize in standard functional analysis since the functional derivatives of (2.15) would act at the arguments $C_{\mu k}(t_k)$ of these functions as well as at the derivatives $\dot{C}_\mu$ in the line elements $dx_\mu = \dot{C}_\mu dt$ The loop calculus is designed in such a way as to fully utilize the independence of the coefficient functions on the form of the loop. The functionals which can be represented in this form will be refered to as Stokes type functionals. The representation (2.15) can be viewed as the loop space analogue of the Taylor expansion.

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} F^n(0) \tag{2.16}$$

In the case of Taylor expansions there is the corresponding identity.

$$F(x+y) = exp\left(x \frac{d}{dy}\right) f(y) \tag{2.17}$$

with an explicit expression for the shift operator. Something like that can be derived in loop space. Namely the following identity is proven in appendix C

$$\phi(C\Gamma) = T exp\left(\int_C dx_\mu \hat{D}^\mu(x)\right) \phi(\Gamma) \tag{2.18}$$

Here $\hat{D}_\mu(x)$ is some nontrivial operator in loop space, depending on area and path derivatives. These derivatives will be defined and discussed at length below. At the moment it is important to realize that the operators $\hat{D}_\mu(x)$ do not depend on the loop C. This loop enters only as an integration domain. By setting Γ to zero, i.e. to the infinitesimal loop, and expanding the exponential we arrive at the representation (2.15) with the coefficient functions.

$$F(X_1, \ldots X_n) = D_{\mu_1}(x_1) \cdots D_{\mu_n}(x_n) \phi(\Gamma) \tag{2.19}$$

Here in what follows we denote the infinitesimal loop as I rather than as O, since it corresponds to the unit loop product U. The relation (2.19) is the loop space analogue of the relation between Taylor coefficient and derivatives of the function. Note that the straightforward Taylor expansion of the functional

$$\phi\left(X_\mu(t)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \ldots \int dt_n \; \times \tag{2.20}$$
$$\times \; X_{\mu_1}(t_1) \ldots\ldots X_{\mu_n}(t_n) \frac{\delta^n \phi}{\delta X_{\mu_1} \ldots \delta X_{\mu_n}}$$

would not be parametric invariant. So it would not be a Taylor expansion on loop space. Since the functions $X_\mu(\cdot)$ differing by representation are identified in the loop space, but not in (2.20).

Area Derivative

Now we start investigating infinitesimal variations of the loop. The general variations reduce to the repeated addition of little closed loops. This statement will become clearer below. Let us first investigate the variation arising when one little closed loop is added. By this we mean that the loop C is replaced by the product loop $C_{xx} \; \tilde{C}_{xx}$ defined as follows.

$$C_{xx} : \; X_\mu = C_\mu(t) \qquad C(0) = C(T) = X \tag{2.21}$$

$$\tilde{C}_{xx} : \; X_\mu = \tilde{C}_\mu(t) \qquad C(0) = ((\tilde{T}) = X \tag{2.22}$$

$$C_{xx}\tilde{C}_{xx} : \quad X = \begin{cases} C_\mu(t) & 0 \le t \le t^T \\ \tilde{C}_\mu(t) & T \le t \le T + \tilde{T} \end{cases} \tag{2.23}$$

In other words, the product loop $C\tilde{C}$ starts as C_{xx}, continues as $\tilde{C}_{xx}$ at the common point and then returns to the same point. This is a family of periodic functions with two periods. Note that in general the product loop contains cusps at x even at the loops $C, \tilde{C}$ were smooth.

Such cusps produce infinities in the ordinary functional derivatives but not in the area derivative. The latter is defined as the leading part of the variation of the functional

$$F(C_{xx}\tilde{C}_{xx}) - F(C_{xx}) = \sigma_{\mu\nu}(\tilde{C})\frac{\delta F}{\delta\sigma_{\mu\nu}(x)} \tag{2.24}$$

The area element $\sigma_{\mu\nu}(\tilde{C})$ is defined as follows

$$\sigma_{\mu\nu}(\tilde{C}) = \frac{1}{2}\int_{\tilde{C}} dy_\nu y_\mu . \tag{2.25}$$

This is the second order invariant. The first order

$$\int_{\tilde{C}} dy_\nu = 0 \tag{2.26}$$

vanishes for the closed loop. In the general case there would be higher order invariants like

$$\int_{\tilde{C}} dy_\mu y_\nu y_\lambda \tag{2.27}$$

but at the infinitesimal loop $\tilde{C}$ we are left with the area element. The area element is antisymmetric

$$\delta\sigma_{\mu\nu} + \delta\sigma_{\nu\mu} = \frac{1}{2}\int_{\tilde{C}} y_\mu dy_\nu + y_\nu dy_\mu = \frac{1}{2}\int_{\tilde{C}} d(y_\mu y_\nu) = 0 \tag{2.28}$$

Therefore the symmetric part of the tensor $\frac{\delta F}{\delta\sigma_{\mu\nu}}$ is not fixed in (2.24). By definition the area is an antisymmetric tensor satisfying (2.24). By taking $\tilde{C}$ as the little square in various planes we may determine all the components of the area derivative. So this definition is unique.

Mandelstam Formula

Let us calculate area derivatives of various functionals to get used to this notion. First of all we find Mandelstam relation

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)} \operatorname{Tr} U(C_{xx}) = \operatorname{Tr} F_{\mu\nu}(x)\, U(C_{xx}) \tag{2.29}$$

This can be easily derived in the Schwinger gauge

$$(y_\mu - x_\mu)\, A_\mu(y) = 0$$

This gauge implies the polar coordinates with origin x. The radial component of the gauge potential vanishes. The linear term in $A_\mu(x)$ near the origin is related to the field strength in Schwinger gauge

$$A_\nu(y) = \frac{1}{2} F_{\mu\nu}(y_\mu - x_\mu) + O(y-x)^2 \tag{2.30}$$

This gauge is analogous to Rieman's normal co-ordinates in curved space. The relation (2.30) can be easily checked from the definition

$$F_{\mu\nu}(y) = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} + [A_\mu, A_\nu] \tag{2.31}$$

Only linear terms contribute at $y = x$. For the higher terms in (2.30) see appendix C. We do not need them here. All we need to obtain the Mandelstam relation is to expand

$$U(C\tilde{C}) = (1 + \int_{\tilde{C}} A_\mu dx^\mu)\, U(C)$$

Area Derivative of Length

As a next exercise let us calculate the area derivative of the length of the loop

$$|C| = \int_C \sqrt{dx_\mu^2} \tag{2.32}$$

Formally the area derivative does not exist since the length of the little loop C is not proportional to the area element. However there is a class of regularized definitions of the length for which the area derivatives exist and do not depend on the form of regularization. Namely

$$|C|_{Reg} = \int_C dx_\mu \int_C d\tilde{x}_\mu \Lambda F\left(\Lambda^2 (x-\tilde{x})^2\right) \tag{2.33}$$

where the regulator function is normalized as follows

$$\int_{-\infty}^{+\infty} d\xi\, F(\xi^2) = 1 \tag{2.34}$$

For the smooth loop at $\Lambda \to \infty$ one may expand

$$\tilde{x} - x = (\tilde{t}-t)\dot{C}_\mu(t) + \ddot{C}_\mu(t)(\tilde{t}-t)^2/2 + \cdots \tag{2.35}$$

$$d\tilde{x} = d(\tilde{t}-t)\left[\dot{C}_\mu(t) + \ddot{C}_\mu(t)(\tilde{t}-t) + \cdots\right] \tag{2.36}$$

$$dx_\mu = dt\, C_\mu(t) \tag{2.37}$$

The conventional parametrization here is the proper length as t, i.e., the constraint.

$$\dot{C}_\mu^2(t) = 1 \tag{2.38}$$

One may easily check that in virtue of (2.34) the regularized definition of length reduces to the standard one at $\Lambda \to \infty$

for the smooth loop.

In the case of a self-intersecting point there would be an additional contribution, which however, tends to zero as Λ^{-1}. This is so, since both x and $\tilde{x}$ vary in the vicinity $O(\Lambda^{-1})$ of the self-intersecting point.

With the regular point only $x-\tilde{x}$ is $O(\Lambda^{-1})$ so the contribution is finite. Only in the case of a self-intersection will the regularized definition differ from the standard one.

As for the area derivative of the regularized length, it is calculated without problems, since this is a Stokes type integral. We find from the ordinary Taylor expansion

$$\delta|C|_{Reg} = \delta \int_{\tilde{C}} d\tilde{x}_\mu \int_C dx_\mu \Lambda F(\Lambda^2(x-\tilde{x})^2) =$$

$$= 2\int_{\tilde{C}} d\tilde{x}_\mu (x_\nu - \tilde{x}_\nu)\Lambda \int_C dx_\mu \frac{\partial}{\partial x_\nu} F(\Lambda^2(x-x^2)^2) \quad (2.39)$$

$$= 4\,\delta\sigma_{\mu\nu}(\tilde{C})\Lambda \int_C dx_\mu \frac{\partial}{\partial x_\nu} F(\Lambda^2(x-\tilde{x})^2)$$

The calculation at $\Lambda \to \infty$ for a smooth loop proceeds as follows (caret $\sqcup$ stands for antisymmetrization)

$$\frac{\delta|C|_{Reg}}{\delta\sigma_{\mu\nu}} = 2\Lambda \int_C \underbrace{dx_\mu \frac{\partial}{\partial x_\nu}}_{} F(\Lambda^2(x-x_0)^2)$$

$$= 4\Lambda^3 \int_C \underbrace{dx_\mu (x-x_0)_\nu}_{} F'(\Lambda^2(x-x_0)^2) \to$$

$$= 4\Lambda^3 \int_{-\infty}^{+\infty} dt(\dot{C}_0 + t\ddot{C}_0)_\mu (\dot{C}_0 t + \tfrac{1}{2}\ddot{C}_0 t^2)_\nu F'(\Lambda^2 t^2) \quad (2.40)$$

$$= 4\Lambda^3 \tfrac{1}{2} (\underbrace{\ddot{C}_\mu(t_0)\dot{C}_\nu(t_0)}_{}) \int_{-\infty}^{+\infty} dt\, t^2 F'(\Lambda^2 t^2)$$

$$= \underbrace{\ddot{C}_\mu(t_0)\dot{C}_\nu(t_0)}_{} \cdot \Lambda \int dt\, t \frac{d}{dt} F(\Lambda^2 t^2) = \underbrace{\ddot{C}_\mu(t_0)\dot{C}_\nu(t_0)}_{}$$

Integration by parts using the normalization condition was used in the last line.

So for the smooth curve the area derivative of length reduces to the orientation tensor in the local tangent plane.

Area Derivative of the General Ansatz

Finally, let us calculate the area derivative of the general ansatz (2.15). By definition we have to replace C by $C_{xx}\tilde{C}_{xx}$ in the term. How the points $x_1 \dots x_n$ are ordered along $\tilde{C}\,C$ produces additional contributions. We have to consider two possibilities:

i) One of the integration points x_i varies inside $\tilde{C}$;

ii) Two adjacent integration points vary inside $\tilde{C}$.

The contribution from three and more points inside $\tilde{C}$ would yield cubic and higher terms in its size. We are interested in the quadratic terms $\sim \delta\sigma_{\mu\nu}(\tilde{C})$. The linear terms (2.26) vanish identically as explained before. Naturally, in the contribution from our integration point we have to expand in $(x - x_0)$ to the linear term, in order to obtain $\delta\sigma_{\mu\nu}$

In the contribution from two adjacent points we replace the integral by it's value at x_0 which would yield

$$\int_{\tilde{C}_{00}} dx_\mu^i \int_{\tilde{C}_{i0}} dx_\nu^{i+1} = \int_{\tilde{C}} dx_\mu^i \, x_\nu^i = 2\,\delta\sigma_{\mu\nu}(\tilde{C}) \tag{2.41}$$

An explicit contribution from these two alternatives lead in the n^{th} term to

$$\int_{C_{00}} dx_2^{\mu_2} \int_{C_{20}} dx_3^{\mu_3} \int_{C_{n-1,0}} dx_n^{\mu_n} T_{\mu\nu}^{\mu_1}(x_0) F(\mu_1 x_0; \mu_2 x_2; \dots \mu_n, x_n) \tag{2.42}$$

with the vertex

$$\hat{T}_{\mu\nu}^{\;\mu}(x) = \delta_{\nu\mu}\frac{\partial}{\partial x_{\mu}} - \delta_{\mu\mu}\frac{\partial}{\partial x^{\nu}} \tag{2.43}$$

and

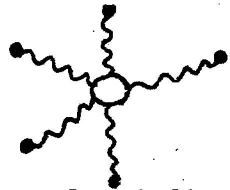
$$\int\limits_{C_{oo}} dx_3^{\mu_3} \ldots \int\limits_{C_{n-1,o}} dx_n^{\mu_n}\, \hat{T}_{\mu\;\nu}^{\;\mu_1\mu_2} F(\omega, x_o; \mu_2 x_o \ldots \mu_n x_n) \tag{2.44}$$

with the vertex

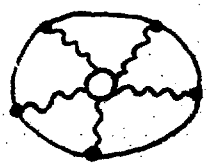
$$\hat{T}_{\mu\;\nu}^{\;\mu_1\mu_2} = \delta_{\mu}^{\mu_1}\delta_{\nu}^{\mu_2} - \delta_{\nu}^{\mu_1}\delta_{\mu}^{\mu_2} \tag{2.45}$$

At a first glance we have to multiply these contributions by the number n of points, but it is not so. Remember there was an arbitrary point x - the origin of parametrization of the loop. When the little loop $\tilde{C}_{x_o x_o}$ is added, a natural origin appears, so the points $x_1 \ldots x_n$ are now ordered with respect to the joining point x_o. The naive insertion of the little loop in n loop integrals would lead to overcounting. By the way, this insertion is formally forbidden in our notations, since only the point x, varies at $\tilde{C}oo$, the rest of the points vary at the remaining pieces $C_{,o}$.

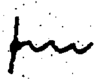
Let us return to our expression for the area derivatives. It is inconvenient to carry all the indices and arguments, so we adopt the graphic notations, the same as in gauge theory. The loop C will be denoted by the oriented solid line, like the quark propagator, but no factors will be associated with this line. The coefficient functions will be denoted as the gluon Green's functions

$$F(\mu_1 x_1 \ldots \mu_n x_n) \quad = $$  $$(2.46)$$

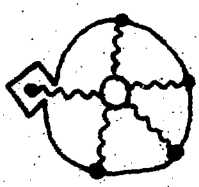
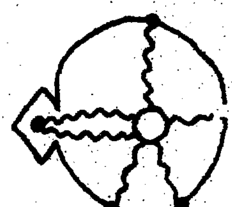
Note however that they do not carry any color indices, and are only cyclic symmetric. The ansatz (2.15) in this not action becomes

$$F = \sum$$  $$(2.47)$$

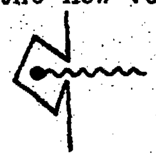
The vertices here correspond to the line elements

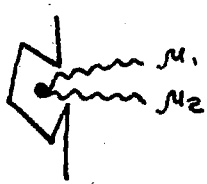
 $$= dx_\mu \qquad (2.48)$$

The area derivative in the same notations can be written as follows

$$\frac{\delta F}{\delta \sigma_{\mu\nu}} = \sum_n$$  $$+ \sum_n$$  $$(2.49)$$

where the new vertices stand for $\hat{T}^{\mu_1}_{\mu\nu}$ and $\hat{T}^{\mu_1,\mu_2}_{\mu\ \nu}$

 $$\mu_1 = T^{\mu_1}_{\mu\nu} \qquad (2.50)$$

 $$= T^{\mu_1,\mu_2}_{\mu\ \nu} \qquad (2.51)$$

This generalization of the Mandelstam relation (2.29) for an art ary functional of the Stokes type. In the case of the loop field these vertices were generated by the linear and quadratic terms in the standard formula (2.31) for the field strength It is important here that these vertices arose without any reference to gauge theory. This is one of the basic elements in reconstructing the loop diagram technique in loop dynamics.

Area Derivatives in Terms of Functional Derivatives

Let us establish the correspondence between the area derivative and the ordinary functional derivatives in cases where the latter exist. Consider the definition (2.29) of the area derivative and let us find the formal functional derivative of both sides with respect to some point $x = \tilde{C}(\tau)$ belonging to the little loop $\tilde{C}$.

Only the first term on the l.h. side contributes and on the r.h. side we have to differentiate $\sigma_{\mu\nu}(\tau)$ in (2.25) There are terms

$$\delta\sigma_{\mu\nu}(\tilde{C}) = \frac{1}{2}\int \delta C_\mu(\tau)\,\dot{C}_\nu\,d\tau + \frac{1}{2}\int C_\mu(\tau)\delta\dot{C}_\nu\,d\tau$$

after integration by parts we find (2.52)

$$\frac{\delta\sigma_{\mu\nu}(\tilde{C})}{\delta\tilde{C}_\alpha(\tau)} = \frac{1}{2}\left(\delta_{\mu\alpha}\dot{C}_\nu - \delta_{\nu\alpha}\dot{C}_\mu\right) \tag{2.53}$$

Therefore

$$\frac{\delta F(c)}{\delta C_\alpha(\tau)} = \frac{\delta F}{\delta\sigma_{\alpha\beta}}\dot{C}_\beta \tag{2.54}$$

for the loop field the corresponding relation is well known. This relation in the general case may be used to define the area derivative in terms of the ordinary functional derivatives. Namely let us find one more functional derivative of (2.54) and take the antisymmetric tensor part. We find

$$\frac{\delta^2 F}{\delta C_\alpha(t)\,\delta C_\beta(t')} = \dot{\delta}(t-t')\,\frac{\delta F}{\delta \sigma_{\alpha\beta}} \tag{2.55}$$

Integrating over the small interval $(t-t')$ with proper weight $(t-t')$ we arrive at Polyakov's definition of the area derivative

$$\frac{\delta F}{\delta \sigma_{\alpha\beta}} = \lim_{\epsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} d\tau\,\tau\,\frac{\delta^2 F}{\delta x_\alpha(T+\tau/2)\,\delta x_\beta(T-\tau/2)} \tag{2.56}$$

This definition has advantages when the functional derivatives can be calculated directly, say in the case of length/c/ One may check the relation (2.40) using this definition. Note that the area derivative satisfies the Leibnitz rule.

$$\frac{\delta(AB)}{\delta \sigma_{\mu\nu}} = \frac{\delta A}{\delta \sigma_{\mu\nu}}\,B + A\,\frac{\delta B}{\delta \sigma_{\mu\nu}} \tag{2.57}$$

as it follows from our definition.

With Polyakov's definition the Leibnitz rule arizes due to cancellations in the integral

$$0 = \int_{-\varepsilon}^{+\varepsilon} \tau\,d\tau\,\left(\frac{\delta A}{\delta x_\alpha(t-T/2)}\,\frac{\delta B}{\delta x_\beta(t+T/2)} + \frac{\delta A}{\delta x_\alpha(t+T/2)}\,\frac{\delta B}{\delta x_\beta(t-T/2)}\right) \tag{2.58}$$

Note that this definition lacks manifest parametric invariance. It is expected to be recovered only at $\varepsilon = 0$. As we know, in gauge theories it is dangerous to use a regularization violating gauge invariance. Some anomalies may be lost in the absence of a gauge invariant limiting procedure. This is why we prefer the original definition which is always correct.

Now we may redefine the Stokes type functional as a parametric invariant functional possessing an arbitrary number of area derivatives $\frac{\delta}{\delta\sigma} \cdots \frac{\delta}{\delta\sigma_n}$ The area derivatives have to be symmetric, as ordinary derivatives

$$\frac{\delta}{\delta\sigma_{\alpha\beta}(y)}\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\Phi = \frac{\delta}{\delta\sigma_{\mu\nu}(x)}\frac{\delta}{\delta\sigma_{\alpha\beta}(y)}\Phi \tag{2.59}$$

This simply means that little loops $\tilde{C}_1$ and $\tilde{C}_2$ at two different points x_1, x_2 can be added in any order, first C_1 and second C_2 or vice versa. At the same time the points cannot be interchanged, since they are ordered along the loop, therefore

$$\frac{\delta}{\delta\sigma_{\alpha\beta}(x-0)}\frac{\delta}{\delta\sigma_{\mu\nu}(x+0)}\Phi \neq \frac{\delta}{\delta\sigma_{\alpha\beta}(x+0)}\frac{\delta}{\delta\sigma_{\mu\nu}(x-0)}\Phi \tag{2.60}$$

In the case of the loop field the difference between the L.H.S. and R.H.S. reduces to the commutator of the field strength. An important identity (Taylor formula in loop space) proven in appendix C demonstrates the equivalence of the new definition of the Stokes type functional and the old one. To virtue of this identity the coefficient functions can be explicitly expressed in terms of the area derivative of the functional, calculated for the some standard loop (the bicycle wheel). We are not yet prepared to discuss this formula, since it involves path derivatives.

Path Derivatives

The definition of the path derivative is even more simple than that of area derivative but it takes time to get used to this notion, because of a certain psychological barrier.

For readers familiar with Jackiws translations accompanied by gauge transformations we may say in advance, that the path derivative represents the loop spece analogue of this notion. We shall not exploit this analogy, but rather start from first principles.

Consider first the function $F(\Gamma_{xy})$ of the open path Γ_{xy} and let us add a little path $\tilde{\Gamma}_{yz}$ to the end point y. The variation of the functional in this case may start from the linear term

$$F(\Gamma_{xy}\Gamma^{z}_{yz}) - \Gamma(\Gamma_{xy}) = (z_\alpha - y_\alpha)\partial^\alpha_y F(\Gamma_{xy}) \,. \tag{2.61}$$

To higher order in the size of the little path $\tilde{\Gamma}$, there would arize like $\int \xi_\alpha d\xi_\beta$ depending on the form of $\tilde{\Gamma}$ but at infinitesimal $\tilde{\Gamma}_{yz}$ we are left with the universal term

$$\int_{\tilde{\Gamma}_{yz}} d\xi_\alpha = z_\alpha - y_\alpha \tag{2.62}$$

Naturally, this expansion exists for a limited class of functionals For some singular functionals the variation may start from terms like $\int_{\tilde{\Gamma}} \sqrt{d\xi^2} : |\tilde{\Gamma}|$ etc. Such functionals should be regularized prior to differentialtion. By regularization we mean the representation in terms of ordered multiple line integrals, as before. With such an ansatz the path derivatives exist. Take an example of the path field

$$\mathcal{U}(\Gamma_{xy}) = P\exp\left(\int_{\Gamma_{xy}} A^\mu dx_\mu\right) \tag{2.63}$$

Direct application of the definition (2.61) yields in this case

$$\partial^{\alpha}(y)\, \mathcal{U}(\Gamma_{xy}) = \mathcal{U}(\Gamma_{xy})\, A^{\alpha}(y) \tag{2.64}$$

In the same way

$$\partial^{\alpha}(x)\, \mathcal{U}(\Gamma_{xy}) = -A^{\alpha}(x)\, \mathcal{U}(\Gamma_{xy}) \tag{2.65}$$

The original definition (2.61) tells us nothing about commutators of these derivatives. In general they do not commute. Consider the path derivative of (2.64). Applying the definition (2.61) once more we find

$$\partial^{\beta}(y)\partial^{\alpha}(y)\, \mathcal{U}(\Gamma_{xy}) = \mathcal{U}(\Gamma_{xy})\left(A^{\beta}(y)A^{\alpha}(y) + \frac{\partial A^{\alpha}(y)}{\partial y_{\beta}}\right) \tag{2.66}$$

The commutator reduces to the field strength

$$[\partial^{\beta}(y), \partial^{\alpha}(y)] = \mathcal{U}(\Gamma_{xy})\, F^{\alpha\beta}(y) \tag{2.67}$$

This is a manifestation of the general rule

$$[\partial^{\beta}(y), \partial^{\alpha}(y)]\, \mathcal{U}(\Gamma_{xy}) = \frac{\delta \mathcal{U}}{\delta \sigma_{\alpha\beta}(y)}(\Gamma_{xy}) \tag{2.68}$$

The commutator reduces to the addition of a closed loop at the end point. We leave to the reader the formal proof based on the definitions of the area and path derivatives.

Relation to Functional Derivatives

The psychological barrier we mentioned above is related to the fact that for analytic curves-straight lines, parabolas etc., another definition seems more natural. Namely, one may consider in the spirit of Hamilton-Jacobi equation the point y or x

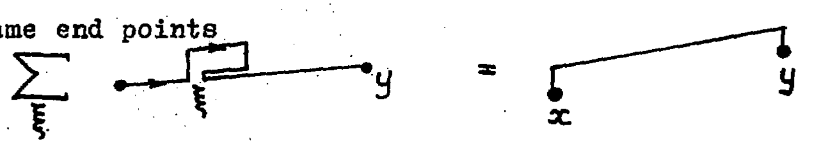
as a parameter in the equation of the curve Γ_{xy}.
Say for the strength line L_{xy}

$$L_{xy}\ ; \quad \xi_\alpha = x_\alpha + (y_\alpha - x_\alpha)\, t/T \tag{2.69}$$

One may differentiate with respect to this parameter without changing the equation of Γ_{xy} so to say, draw the same curve from x to $y + \delta y$. In the general case we do not know, how the equation of Γ_{xy} involves the parameters x, y moreover, it may be nonanalytic so that only an infinitesimal part of Γ_{xy} will change when y will change to $y+\delta y$. To save the definition one may consider, say, the rotation of the curve around x or rescaling etc. All such transformations reduce to contribution of area derivatives and our path derivatives. Namely, the general variation of $F(\Gamma)$ reads

$$\delta F = \int_{\Gamma_{xy}} d\xi_\nu \delta \xi_\mu \frac{\delta F(\Gamma)}{\delta \sigma_{\mu\nu}(\xi)} + \delta y_\mu \partial^\mu(y) F(\Gamma_{xy}) + \delta x_\mu \partial^\mu(x) F(\Gamma_{xy}) \tag{2.70}$$

In the same way, as with the area derivatives before one may check the equivalence of the formula to the previous definition. We leave this exercise to the reader. The meaning of (2.70) is obvious. The body of the path Γ_{xy} is shifted by the integral with the area derivative, and the end points are shifted by the path derivatives. The last step is necessary, since the area derivatives replace Γ_{xy} by the Γ'_{xy} with the same end points



(2.71)

Relation (2.70) is quite useful. By taking various $\delta\xi_\mu$ corresponding to translation ($\delta\xi_\mu$ =const), rotations $(\delta\xi_\mu = \omega_{\mu\nu}\xi^\nu)$, dilations $(\delta\xi_\mu = \lambda\xi_\mu)$ and special conformal transformations $(\delta\xi_\mu = (2\xi_\mu\xi_\nu - \xi^2\delta_{\mu\nu})\lambda^\nu)$ one may derive various Ward identities for the loop functionals in quantum theory We'll return to these identities later on. At the moment let us note the relations

$$\partial^\mu(y)\,F(\Gamma_{xy}) = \int_{T-\varepsilon}^{T} dt\; \frac{\delta F}{\delta\Gamma_\mu(t)} \tag{2.72}$$

$$\partial^\mu(x)\,F(\Gamma_{xy}) = -\int_0^{\varepsilon} dt\; \frac{\delta F}{\delta\Gamma_\mu(t)} \tag{2.73}$$

which follow from these identities. As in the case of the area derivatives the relations may serve as formal definitions of the path derivatives with a bit of scepticism concerning the parametric invariance. The two definitions can be compared in the case of length.

$$|\Gamma_{xy}| = \int_{\Gamma_{xy}} \sqrt{dx^2} = \int_0^T dt\,\sqrt{\dot{\Gamma}_\mu^2} \tag{2.74}$$

According to (2.72), (2.73)

$$\partial^\mu(y)\,|\Gamma_{xy}| = \dot{\Gamma}_\mu(T-0)\,/\sqrt{\dot{\Gamma}^2} \tag{2.75}$$

$$\partial^\mu(x)\,|\Gamma_{xy}| = -\dot{\Gamma}_\mu(+0)\,/\sqrt{\dot{\Gamma}^2} \tag{2.76}$$

The same can be found from the regularized definition

$$|\Gamma_{xy}| = \Lambda \int_{\Gamma_{xy}} d\xi_\mu \int_{\Gamma_{xy}} d\xi_\mu\;\; \Gamma\left(\Lambda^2(\xi-\xi)^2\right) \tag{2.77}$$

by applying (2.61). We omit these trivial calculations. The reader is advised to do them himself to understand better the meaning of path derivative in the case of length, as we see, it yields the tangent vector at the end point. This means that adding the little path $\tilde{\Gamma}_{yz}$ in any other transverse direction would yield zero. Only the path $\tilde{\tilde{\Gamma}}_{yz}$ continuing the original will contribute. However, this is not a general rule, as we have seen by an example of the path field (2.63). There all the little paths including the ones orthogonals to Γ led to a nonvanishing contribution (one more reason to include irregular paths).

Shift of a Marked Point

As a next step let us consider a closed loop C_{xx} and apply the path derivative to some functional depending on C_{xx}. Here the point acts as a beginning as well as the end of the loop, so it should be shifted twice

$$F(\tilde{\Gamma}_{zx} C_{xx} \tilde{\tilde{\Gamma}}_{xz}) - F(C_{xx}) = (z_\alpha - x_\alpha)\partial^\alpha(x) F \qquad (2.78)$$

Take as an example the path field (2.63). In this case

$$\partial^\alpha(x)\, U(C_{xx}) = [U(C_{xx}), A^\alpha(x)] \qquad (2.79)$$

As for the Wilson Loop after taking the trace, the path derivative vanishes

$$\partial^\alpha(x)\, \mathrm{tr}\; U(C_{xx}) = 0 \qquad (2.80)$$

This is a general statement. Whenever we shift the point of the loop in Stokes type functionals, it can be regarded as adding a little closed loop $\widetilde{C}_{xx}$. In the above formula(2.78) this loop consisted of two wires $\widetilde{\widetilde{\Gamma}}_{x}$ and $\widetilde{\Gamma}_{zx}$ i.e. $\widetilde{C}_{xx} = \widetilde{\widetilde{\Gamma}}_{xz} \widetilde{\Gamma}_{zx}$ For the Stokes type functionals F(c) the variation starts from the quadratic term $\delta\sigma_{\mu\nu}(\widetilde{C})$, hence the coefficient of $\partial^{\alpha} F(C)$ at the linear term must vanish. So for the Stokes type functional

$$\partial^{\alpha}(x)\,\phi(c) = 0 \qquad (2.\ 81)$$

This is so if x is not a marked point. As it was in the case of the path field before taking the trace. The Stokes type functional by definition does not possess such marked points, but they may arize after some limiting procedure.

The simplest and at the same time the most important example is the marked point produced by applying the area derivative. The area derivative in Stokes type functionals is not the Stokes type functional, since additional area derivatives cannot be defined at the point where one derivative was al - ready applied. The limits from the left and from the right in the case of the Wilson loop, for example, differ by the commutator of the field strength. We refer to such functionals as Stokes type functionals with marked points, since everywhere but at the marked point the area derivative makes sense. the generalization to Stokes type functionals with several marked points is obvious.

In order to see how the relation (2.81) works for Stokes type functionals and how it breaks at the marked point, let us consider another example. Take the length of the loop C_{xx} which in general is not a Stokes type functional. The area derivative calculated above involved the second derivative $\ddot{C}(t)$ of the equation of the curve i.e the curvature. At the cusp, where the tangent c is discontinous, the area derivative does not exist. So, the cusp is a marked point for the length. Let us calculate the path derivative. Combining above formulas we find (with t being the length)

$$\partial^{\alpha}(x)\,|\,C_{xx}\,| = \dot{C}_{\alpha}(t-0) - \dot{C}_{\alpha}(t+0) \tag{2.82}$$

In agreement with above arguments the path derivative vanishes, when the area derivative exists. Let us now apply the path derivative to the Mandelstam relation. Simple algebra yields.

$$\gamma^{\alpha}(x)\,\frac{\delta\Phi}{\delta\sigma_{\mu\nu}(x)} = \frac{tr}{N}\left(\nabla_{\alpha} F_{\mu\nu}(x)\,\mathcal{U}\left(C_{xx}\right)\right) \tag{2.83}$$

where

$$\nabla_{\alpha} = \frac{\partial}{\partial x_{\alpha}} + [\,A_{\alpha}\,, \tag{2.84}$$

is the covariant derivative in the adjoint representation.

We come to the heart of the matter. It turns out possible to insert the field strength and it's covariant derivatives inside colour traces by purely colourless agents. We perform certain geometric manipulations, add some loops and shift them aside and we end with the same objects which enter the Yang-Mills equations. This reveals some new geometric meaning of these equations. Due to this property it is possible to reformulate gauge theory as loop dynamics.

Let us investigate the identity (2.83) in some detail. In virtue of the Bianchi identity

$$\nabla_\alpha F_{\mu\nu} + \nabla_\mu F_{\nu\alpha} + \nabla_\nu F_{\alpha\mu} = 0 \tag{2.85}$$

We may write the identity

$$\partial^\alpha(x)\frac{\delta\Phi(C)}{\delta\sigma_{\mu\nu}} + \partial^\mu(x)\frac{\delta\Phi(C)}{\delta\sigma_{\nu\alpha}} + \partial^\nu(x)\frac{\delta\Phi(C)}{\delta\sigma_{\alpha\mu}} = 0 \tag{2.86}$$

The geometric meaning of the operator on the left hand side is as follows. Introduce the little closed 2-surface δS_x at the point $x \in C$ and the little loop $\tilde{C}_{yy}$ on this surface connected with C_{xx} by the little wires Γ_{xy} Γ'_{yx}. After that let the little loop vary at the little surface, with the wires following point y . We obtain the integral

$$\int_{\delta S} d\sigma_{\mu\nu}(y)\ \frac{\delta\Phi}{\delta\sigma_{\mu\nu}(y)}\ (C_{xx}\,\Gamma_{xy}\,\Gamma'_{y\cdot x}) = 0 \tag{2.87}$$

It vanishes since this is the integral over a closed surface of a "total derivative". In mathematical language it vanishes since $dd = 0$

Expanding (2.87) in $(y-x)$ to the linear terms we arrive at the identity (2.86). The zeroth terms in $y-x$ vanish identically, since the surface is closed. The above proof shows that this is a general relation, which holds for an arbitrary functional, not only for the loop field. This can also be checked by means of the Taylor formula in appendix C.

Another useful identity

$$\partial^\alpha(x)\,\partial^\beta(x)\,\frac{\delta\Phi}{\delta\sigma_{\alpha\beta}(x)} = 0 \tag{2.88}$$

follows for the wilson loop from the identity

$$\nabla^\alpha\nabla^\beta F_{\alpha\beta} = \frac{1}{2}\left[F_{\alpha\beta}F^{\alpha\beta}\right] = 0 \tag{2.89}$$

In the general case it follows the identity

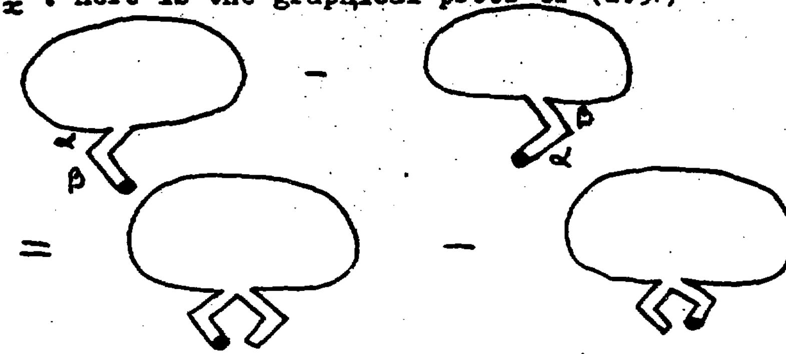
$$\partial^{\alpha}(x)\partial^{\beta}(x)\frac{\delta\Phi}{\delta\sigma_{\alpha\beta(x)}} = \frac{1}{2}[\partial^{\alpha}(x)\partial^{\beta}(x)]\frac{\delta\Phi}{\delta\sigma_{\alpha\beta(x)}} =$$

$$= \frac{1}{2}\left(\frac{\delta}{\delta\sigma_{\alpha\beta(x+0)}} - \frac{\delta}{\delta\sigma_{\alpha\beta(x-0)}}\right)\frac{\delta\Phi}{\delta\sigma_{\alpha\beta(x)}} = 0 \qquad (2.90)$$

The last line was derived from the relation

$$[\gamma^{\alpha}\gamma^{\beta}]\Phi(C_{xx}) = \frac{\delta\Phi}{\delta\sigma_{\alpha\beta(x-0)}} - \frac{\delta\Phi}{\delta\sigma_{\alpha\beta(x+0)}} \qquad (2.91)$$

This relation is a direct consequence of the identity(2.68) for the open path. The negative sign comes from the difference in orientation of the beginning $x+0$ and the end $x-0$ of the loop C_{xx}. Here is the graphical proof of (2.91)



(2.92)

On the left hand side we shifted the marked point in the α direction and then in the β direction and the other way around in the second term. On the right hand side we added more wires to the loops which is always possible at the unmarked point. due to (2.81). After that we obtained closed loops. In the first term the closed loop is to the right of the marked point and in the second term it is to the left of the marked point. Comparing the term $\sim dx^{\alpha}\wedge dx^{\beta}$ we arrive at (2.91). Finally note, that the identities (2.86) and (2.91) can also be derived from the formal relations between our derivatives and ordinary functional derivatives, for example, the Bianchi identity (2.86) follows from the symmetry of the third functional derivatives. We insist so much on the geometric meaning of all the operators and relations of the loop calculus, because this geometric meaning is the key to understand the relations between loop dynamics and the string theory.

Conclusions

i) Loop space at $d \neq 0$ dimensions has nontrivial topology. At large N most of the original degrees of freedom of the gauge field is hidden in self-intersecting loops, traversed many times.

The number of independent loop variables grows as N^2 which causes technical difficulties when introducing these fields as collective variables.

ii) The loop fields belong to a special class of functionals- functionals of the Stokes type.

The standard functional derivatives may not exist for Stokes type functionals, but there are special derivatives, the area and path derivative, which always make sense. For the Wilson loop these derivatives insert the field strength and covariant derivative in the adjoint representation inside the colour trace.

iii) The area and path derivatives obey a simple algebra, investigated above and in appendix C. One may calculate these derivatives for arbitrary functionals either from formal relations with the standard functional derivative or (which is sometimes simpler)directly from this algebra and the original definition.

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