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In refs. [1-4] classical equations of motion of non-Abelian gauge fields were investigated, when the vector potential $A_\mu^a(x)$ depends on time only. It is quite natural to call this system the Yang-Mills classical mechanics (YMCM) [1]. In the gauge $A_0^a = 0$ the YMCM Hamiltonian may be written in the form [4]

$$H_{YM} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{g^2}{2}(x^2y^2 + y^2z^2 + z^2x^2) + T_{YM}, \tag{1}$$

$$T_{YM} = \frac{1}{2}\sum_{a,\kappa}(I_\kappa \Omega_\kappa^2 + I_a \omega_a^2 - 2 I_{a\kappa} \Omega_\kappa \omega_a),$$

where $I_1 = y^2 + z^2$, $I_2 = x^2 + z^2$, $I_3 = x^2 + y^2$, $I_{11} = 2yz$,

$I_{22} = 2xz$, $I_{33} = 2xy$, $I_{a\kappa} = 0$ at $a \neq \kappa$. Equations of motion for the rotational degrees of freedom have the form [4]

$$\dot{\vec{M}} + \vec{\Omega} \times \vec{M} = 0, \qquad \dot{\vec{N}} + \vec{\omega} \times \vec{N} = 0, \tag{2}$$

where $M_\kappa = I_\kappa \Omega_\kappa - I_{\kappa a}\omega_a$, $N_a = I_a \omega_a - I_{a\kappa}\Omega_\kappa$. The system (1) has three integrals H_{YM}, M^2, N^2.

We shall call the system with the Hamiltonian

$$H_{FS} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{g^2}{2}(x^2y^2 + y^2z^2 + z^2x^2) \qquad (3)$$

the fundamental subsystem (FS) of the YMCM. It corresponds to the case, when the matrix A_i^a is diagonal, or $N_a = M_K = \omega_a = \rho_K = 0$. The strong instability of the FS trajectories with respect to small changes of initial conditions has drawn the authors of refs. [1,2] to a conclusion on the nonintegrability and stochasticity of the YMCM. It is significant that this system has a countable number of periodical trajectories[1].

The aim of the present paper is to investigate the statistical properties of the YMCM from the viewpoint of ergodic theory [5-7]. Consider first the FS. As has already been mentioned [1,2], trajectories of FS have a strong instability, which leads to the fact that the phase trajectory chaotically fills up the energy surface $H_{FS} = E$. It is intuitively clear what is implied by the chaotic or stochastic motion, but it is necessary to exactly define the measure of this chaos.

In ergodic theory [5-7] dynamical systems (DS) are classified by the degree of increase of chaotic-stochastic properties: they are ergodic systems, systems with weak mixing, with mixing, with n-fold mixing, and, finally, with K-mixing (K-systems). It is important to establish what class of nonintegrable DS the YMCM belongs to. It is also clear that if almost every trajectory of the system fills up the energy surface, then there is no sense in searching for its concrete solutions. Instead one should investigate the global properties of the system as a whole. After establishing the class which the YMCM

belongs to, one may indicate these global properties.

Let us denote the FS solution with the initial condition X_0 by $X_t = T^t X_0$, where X is the point of the phase space M with the coordinates (x, y, z, p_x, p_y, p_z). Since almost every FS trajectory fills up completely the energy surface [1,2], or, in other words, the FS has no other isolating integrals except energy, we obtain that the FS is at the minimum an ergodic system *.

In papers by Hadamard, Hedlund, Hopf, Krylov, Anosov, Sinai et al. [8-13] general enough criteria were obtained, answering the question what DS class the system with the given Hamiltonian H belongs to. It has been proven, in particular, that the geodesic flow on the Riemannian manifolds of the variable of negative curvature is a K-flow.

On the other hand, from the Maupertuis principle it is known, that the DS trajectories are geodesic lines of Riemannian metrics given in the domain of the configuration space $U(q) < E$ ($U(q)$ is the potential and E the total energy of the system)

$$ds^2 = (E - U(q))(dq_1^2 + \dots + dq_N^2) \qquad (4)$$

Consider for simplicity the two-dimensional FS, then

$$ds^2 = \lambda(q_1, q_2)(dq_1^2 + dq_2^2) = \left(E - \frac{g^2}{2}(q_1 q_2)^2\right)(dq_1^2 + dq_2^2) \qquad (5)$$

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*By their definition ergodic systems have no invariant subsets in M of the nonzero measure [5-7]. In the presence of the additional isolating integral $\phi(x)$ the condition $\phi(x)$ = const will single out the invariant subset in M.

and the curvature in the range $U_{FS} < E$ is

$$R = -\frac{1}{2\lambda}\Delta \ln\lambda = \frac{(q_1^2+q_2^2)(E+\frac{g^2}{2}q_1^2 q_2^2)}{(E-\frac{g^2}{2}q_1^2 q_2^2)^3} > 0 \qquad (6)$$

and is strictly positive. The derived conclusion is valid for the $N \leq 6$-dimensional FS as well. Thus, the sufficient condition in this case is not satisfied, and the positivity of R implies, that trajectories are stable, if considered only in the intervals between the scatterings on the equipotential boundary $U_{FS} = E$ on which the metrics (5) has a singularity (see fig. 1 from ref. [1]).

The main statement of the present paper is that the instability of FS develops only due to the scattering on the equipotential boundary $U_{FS} = E$ (figs. 1,2). To make sure in this, let us follow tha analogy between the FS and the scattering billiards of Krylov-Sinai [11-13] with hyperbolic walls, coinciding in their form with the equipotential boundary $U_{FS} = E$ (figs. 1,2).

Consider the following perturbation of the FS

$$U_{FS} \to U_{FS}(\alpha) = \frac{g^2}{2}\left(q_1^\alpha q_2^\alpha + q_2^\alpha q_3^\alpha + q_3^\alpha q_1^\alpha\right) \qquad (7)$$

The parameter α determines the curve in the space of vector fields $(P_i, -\partial U/\partial q_i)$ defining the DS. At $\alpha = 2$ the system (7) coincides with the FS (3) (point A in fig. 3), and at $\alpha = \infty$ (point B in fig. 3) it is the Sinai billiards with absolutely elastic hyperbolic walls in figs. 1,2.

We shall call the DS T^t structurally stable <u>in the narrow sense</u>. if it is structurally stable with respect to perturbations <u>only along some curve</u> in the space of vector fields*.

Fig. 3

Calculating periodical and nonperiodical trajectories of the system (7) with arbitrary α and $\alpha + \Delta\alpha$, we have been convinced that the FS is structurally stable in the narrow sense along the curve defined by the parameter α. This implies that the FS is equivalent to the billiards system of Krylov-Sinai with hyperbolic walls (figs. 1,2).

From this fact follow several important conclusions: first, the strong instability of the FS develops at the scattering of trajectories on the convex inside equipotential boundary of negative curvature; secondly, trajectories of the FS decompose into beams of exponentially compressing and expanding trajectories (the transverse fibers [13]), whose structure is similar to that of Sinai's billiards system with hyperbolic walls ($\alpha = \infty$) and, hence, thirdly, the FS is a

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* The DS T^t is called <u>structurally stable</u>, if for <u>any</u> system S^t <u>in the vicinity</u> of T^t there is a mutually single-valued and continuou transformation, close to the identical one, transforming the trajectories of the system T^t into those of the system S^t, with the direction of the motion being preserved [14].

Kolmogorov K-system and possesses the strongest statistical properties.

Up to now we have discussed two- and three-dimensional FS. Consider now an FS which arises in gauge theories with higher order symmetry groups. Let us define the N-dimensional FS as

$$H^{(N)}_{FS} = \sum_{i=1}^{N} \frac{1}{2}\dot{q}_i^2 + \frac{g^2}{4}\left[(q_1^2+\dots+q_N^2)^2 - q_1^4 - \dots - q_N^4\right]$$

and calculate the curvature in the range of configuration space $U_{FS} < E$. According to (4) the metrical tensor is

$$g_{ij} = (E - U^{(N)}_{FS})\,\delta_{ij} = \lambda(q)\,\delta_{ij}$$

and the scalar curvature is

$$\frac{R^{(N)}_{FS}}{N(N-1)} = -\frac{1}{N\lambda^2}\Delta\lambda - \left(\frac{1}{4} - \frac{3}{2N}\right)\frac{(\nabla\lambda)^2}{\lambda^3} = \frac{(N-1)}{N}\,\frac{(q_1^2+\dots+q_N^2)}{\lambda^2} - \left(\frac{1}{4} - \frac{3}{2N}\right)\frac{\sum_{i=1}^{N} q_i^2(q_1^2+\dots+q_{i-1}^2+q_{i+1}^2+\dots+q_N^2)}{\lambda^3} \qquad (8)$$

It is seen from this expression for curvature that at N 6, $R^{(N)}_{FS} \geq 0$. This implies that within the range $U_{FS} < E$ trajectories are again stable, and the instability arises due to the scattering on the equipotential boundary $U_{FS} = E$ of negative curvature, since the FS is homeomorphic to the scattering hyperbolic billiards. This fact can't be established from (8) since it is valid beyond the equipotential boundary $U_{FS} = E$ only, and doesn't take into account the boundary effects which are essential in this case.

A surprise arising at N 6 is the emergence of the range with equipotential instability defined by

$$\frac{N-1}{N}(E - U_{FS})(q_1^2+\dots+q_N^2) < \left(\frac{1}{4} - \frac{3}{2N}\right)\left[\sum_{i=1}^{N} q_i^2(q_1^2+\dots+q_{i-1}^2+q_{i+1}^2+q_N^2)\right] \qquad (9)$$

since there $R^{(N)}_{FS} < 0$.

From (9) follows that the range of exponential instability is along the equipotential boundary and broadens inside the range $U^{(N)}_{FS} < E$ with $N \to \infty$! This comes to once again confirm the fact that the instability arises due to the equipotential boundary.

The above properties are those of the global characteristics of classical DS. There are two characteristics which are important for the understanding of the YM quantum mechanics and, respectively, of the K-system quantum mechanics.

One may juxtapose a number $h(T^t)$, called the Kolmogorov entropy [15], to each DS T^t. The quantity $h(T^t)$ is positive, and for various DS it takes values from 0 to $+\infty$. K-systems are known to have a positive entropy [15]. Since the FS is a K-system, its entropy is strictly positive

$$h(T^t_{FS}) > 0 \qquad (10)$$

Besides, in 1931 Koopman has proven that to each DS T^t is conjugated a one-parametric group of unitary operators defined by the relation [16]

$$U^t f(x) = f(T^t x) \qquad (11)$$

where $f(x)$ are the complex functions in M quadratically summed ($f(x) \in L_2(M)$). In the case of K-flows one may completely calculate the group U^t spectrum (it is countably multiple Lebesgue spectrum [15]). This implies that the Gilbert space $L_2(M)$ is decomposed into an orthogonal direct sum of the countable set of cyclic subspaces with a simple Lebesgue spectrum, on each of which the group U^t is realized as a group of shift operators

$$U^t f(x) = f(x+t) \tag{12}$$

Thus, the group U^t_{FS} spectrum is completely calculated.

Consider now the YMCM in a more general case, when only the density of external quark currents is zero $N_a = 0$ and $M_K \neq 0$. For such vacuum fields $\omega_a = I_{aK} \Omega_K / I_a$ and the kinetic energy has the form [4]

$$T_{YM} = \sum_K \frac{I_K^2 - I_{KK}^2}{2 I_K} \Omega_K^2 \ , \quad M_K = \frac{I_K^2 - I_{KK}^2}{I_K} \Omega_K. \tag{13}$$

It is convenient to again assume the variables x, y, z to be the coordinates of the material point performing compound motion in the three-dimensional space limited already by effective equipotential surface $U_{eff} = U_{FS} + T_{YM} = E$, and the variables M_K to be the point on a sphere with the radius M^2 = const.

At $M_K = 0\ (\Omega_K = T_{YM} = 0)$ the YMCM coincides with the FS, whose statistical properites are already known. If $M_K \neq 0$ and the variables x, y, z (i.e. the inertia moments I_K, I_a) are not time-dependent, then T_{YM} is conserved together with M^2 and Euler equations (2) for M_K may be explicitly integrated. The point with the coordinates M_K will describe the closed curve on the sphere surface, i.e. the rotational degrees of freedom of M_K have no independent statistical properties.

Let now $M_K \neq 0$, and the variables x, y, z (I_K, I_{Ka}) are time-dependent. In the general case we have two systems: the FS and the gauge gyroscope (13), interacting through inertia moments I_K, I_{Ka}. The effective equipotential surface $U_{eff} = U_{FS} + T_{YM} = E$ lies within the surface $U_{FS} = E$ and is convex inside too, except for the regions near $x = \pm y$, $y = \pm z$, $z = \pm x$. The motion of the point with the coordinates x, y, z is again unstable, and, hence, the rotation of the gyroscope with inertia moments depending on x, y, z will be unstable. The point with the coordinates M_K now moves on the sphere unstably and covers all the compact surface M^2 = const. In other words, the FS induces statistical properties of the YMCM.

The resluts obtained have a theoretically-nonperturbative character, since the classical perturbation theory describes the system behavior during a finite time interval. No statements on the system behavior at $t \to \pm\infty$ can be obtained in any finite order of classical perturbation theory.

In conclusion let us note that the investigation of the YMCM may shed light on the problem of the structure of the QCD vacuum and confinement problem [17-19]. The quantum mechanics of the system (1) should be investigated, which essentially differs from that of the QED harmonic oscillator. In the subsequent paper we shall present the results of the investigation of the YM quantum mechanics essentially basing on the results obtained when considering the YM classical equations from the viewpoint of ergodic theory.

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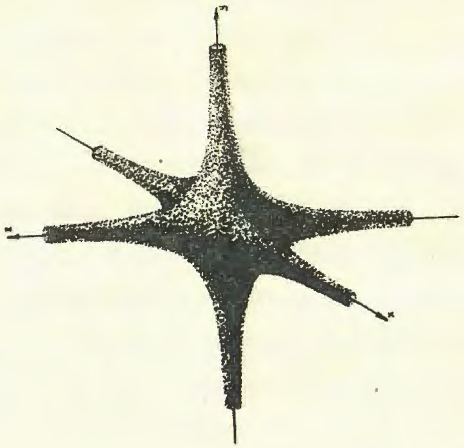

Fig. 2

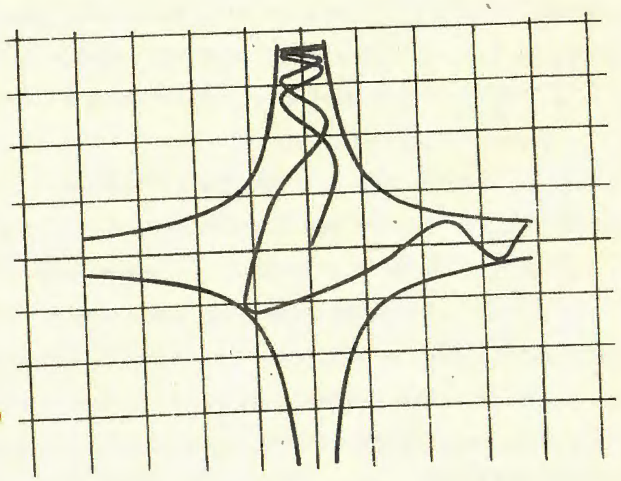

Fig. 1

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