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1/Q - EXPANSIONS OF SPIN AND GAUGE POTTS MODELS. LAGRANGIAN FORMULATION

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I/Q - РАЗЛОЖЕНИЕ СПИНОВЫХ И КАЛИБРОВОЧНЫХ МОДЕЛЕЙ ПОТТСА. ЛАГРАНЖЕВЫЙ ФОРМАЛИЗМ

Проводится I/Q - разложение спиновых и калибровочных моделей Поттса в лагранжевом формализме. При этом существенно используется представление для статистических сумм моделей в терминах теории графов.

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1/Q - EXPANSIONS OF SPIN AND GAUGE POTTS MODELS. LAGRANGIAN FORMULATION

1/Q-expansion for spin and gauge Potts models in the Lagrangian formulation is carried out. The representation of the partition functions in terms of the graph theory language is essentially used.

Yerevan Physics Institute

Yerevan 1983

Y E R E V A N   P H Y S I C S   I N S T I T U T E

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1. Introduction

Spin and gauge Potts models are the natural generalization of the (Izing) spin and gauge models to more-than-two components. Kogut proposed a lattice gauge theory with matter field, which had $Z(Q)$ local symmetry [1]. He introduced two temperature-like variables, K and β, controlling respectively the fluctuations of the gauge and matter fields.

The phase diagram in (β. K) plane is known to have the following properties:

i) the $\beta = 0$ boundary corresponds to the pure gauge theory which undergoes a confining-non-confining transition at some $K = K_c$:

ii) on the $K = \infty$ boundary the theory reduces to a pure spin system which undergoes an order-disorder transition at some β_c .

In this paper we consider only i) and ii) limiting cases.

The quantum Hamiltonian versions of the Q-state Potts models were solved in the 1/Q expansion for an arbitrary dimension d [2] . We have received the 1/Q expansion for the free energy per site of the Potts models in the Lagrangian formulation, and the expansion parameter 1/Q is indenemdent of the dimensions, too.

The analogous problem was solved by P.Ginsparg et al. [3], but they found that the expansion parameter was $1/Q^{1/d}$ which depends on dimension d. $Z(Q)$ -group is the center of the appropriate SU group, and the models with $Z(Q)$ symmetry possess some important features in common with lattice quantum chromodynamics [4]. Moreover, the presence of $Z(Q)$ monopoles plausibly leads to a phase transition [5].

So, a more detailed investigation of the Potts models is of certain interest. In Sec.2 we define the models and make 1/Q expansion for the free energy per site in two dimensions (d=2). Calculating the latent heat we find that it is in good agreement with Baxter's exact result [6]. Partition functions for the gauge Potts models are rewritten in the language of graph theory in Sec.3, just as was done by Baxter for spin models. These representations of the partition functions are useful for 1/Q expansion and closely connected with topological invariants (Betti numbers). In Sec.4 we treat Potts models on a d=3 dimensional lattice.

2. The Two-Dimensional Potts Model.

Potts spin systems [7] are interesting theoretical models of first- and second-order phase transitions. It was shown that two-dimensional Q-component Potts model has a first-order phase transition for $Q > 4$, and a higher-order transition for $Q \leq 4$ [6].

The Q-state Potts model [7] is described by the action

$$S = -\beta \sum_{\langle ij \rangle} \delta_{\sigma_i \cdot \sigma_j} \qquad (2.1)$$

where the site variables σ_i range over the discrete values 0, 1,2,..., Q-1 and the sum in Eq.(1) extends over all nearest neighbours on a d-dimensional lattice.

Write the partition function of the model in the form of

$$Z_G(Q;\beta) = \sum_{spin} \prod_{<ij>} \left(1 + v\delta_{\sigma_i \sigma_j}\right) \tag{2.2}$$

where

$$v = e^{\beta} - 1 \tag{2.3}$$

The partition function (2.2) takes the following simple form after carrying out the spin summations [6] :

$$Z_G(Q;\beta) = \sum_{G' \leq G} v^{b(G')} Q^{n(G')} \tag{2.4}$$

where the summation is over all subgraphs G' (i.e. ways of drawing lines on the edges of the lattice G). $b(G')$ - is the number of lines in G', and $n(G')$ is the number of connected pieces. The partition function, written in the form (2.4), will be useful for 1/Q-expansion independently of dimension.

For a lattice G of N sites, the free energy per site is given by

$$-\beta f = \lim_{N\to\infty} \frac{1}{N} \ln Z_G(Q;\beta, N) = \tag{2.5}$$

$$= \ln Q + \lim_{N\to\infty} \frac{1}{N} \ln\left(\sum_{G'} v^{b(G')} Q^{n(G') - N}\right) \tag{2.5a}$$

The sum under $\ln$ can be written as a series of 1/Q-expansion $(n(G') \leq N)$. The leading terms in such expansion are given by subgraphs G' with $n(G') = N$; $n(G') = N-1$, $n(G') = N-2$, etc. One can easily see that expansion parameter (1/Q) is independent of the dimension d , as it was in the Hamiltonian formulation case. We suppose that only terms proportional to N survive in this 1/Q expansion for the free energy. This fact in the cases d=2 and d=3 will be proved by direct calculations below.

First of all, let us make 1/Q-expansion for the Potts spin model, defined on the two-dimensional square lattice. For this purpose write out the partition function's few terms in 1/Q-expansion

$$Z = Q^N \left\{ 1 + \frac{2N v}{Q} + \frac{N(2N-1) v^2}{Q^2} + \right. \tag{2.6}$$

$$+ \frac{N(2N-1)(2N-2)}{3} \frac{v^3}{Q^3} + N \frac{v^4}{Q^3} + \left(C_{2N}^4 - N \right) \frac{v^4}{Q^4} +$$

$$+ N(2N-4) \frac{v^5}{Q^4} + \left(C_{2N}^5 - 2N^2 + 4N \right) \frac{v^5}{Q^5} +$$

$$+ \left(2N + N C_{2N-4}^2 \right) \frac{v^6}{Q^5} + 2N \frac{v^7}{Q^5} + \dots$$

It follows from (2.6) that only terms proportional to N survive in free energy $1/Q$ expansion (all terms of higher in N orders cancel):

$$-\beta f_{h.t} = \ln Q + \frac{2v}{Q} - \frac{v^2}{Q^2} + \frac{v^3}{Q^3} \left(\frac{2}{3} + v \right) - \tag{2.7}$$

$$- \frac{v^4}{Q^4} \left(\frac{3}{2} + 4v \right) + \frac{v^5}{Q^5} \left(\frac{22}{5} + 12v + 2v^2 \right)$$

In this form the free energy is rather efficient in high-temperature region ($v \ll 1$). For low-temperature region expansion, rewrite the partition function (2.4) in terms of the dual lattice [8]:

$$Z_G(Q,v) = v^E Q^{1-N} \sum_{\mathcal{D}' \subseteq \mathcal{D}} \left(\frac{Q}{v}\right)^{b(\mathcal{D}')} Q^{n(\mathcal{D}')} = \tag{2.8}$$

$$= v^E Q^{1-N} Z_{\mathcal{D}}(Q, v^*)$$

where

$$v^* v = Q \tag{2.9}$$

and $E = b(G') + b(\mathcal{D}') = 2N$, is full number of edges of graph G. This model is self-dual and its transition, if unique, is located at $v_c = \sqrt{Q} \equiv z^{-1}$

The free energy for the low-temperature region can be calculated in the same way as (2.7) using (2.8)

$$-\beta f_{e.t.}(v) = -\ln Q + 2\ln v - \beta f_{h.t}\left(v^* = \frac{Q}{v}\right) \tag{2.10}$$

To calculate the latent heat of the model, one should subtract temperature derivatives of the free energies (2.7) and (2.10) from each other in $v_c = \sqrt{Q} = z^{-1}$

$$L = 1 - 2z - 2z^2 + 4z^3 - 6z^4 + 12z^5 + \dots \qquad (2.11)$$

Formula (2.11) reproduces Baxter's exact result [6] expanded in $z = \frac{1}{\sqrt{Q}}$ series.

We hope that 1/Q-expansion developed in the language of graph theory works in higher dimensions.

3. Graph Theory Language

To develop 1/Q-expansion for Potts spin models we have used graph theory language (2.4). Is it possible to treat gauge Potts models in the same way? We shall show in this section that the answer is "yes".

Consider the Q-state Potts lattice gauge theory

$$S_g = -\frac{K}{Q} \sum_{plaq} \sum_{n=1}^{Q} \left(R_1 R_2 R_3 R_4 \right)^n , \qquad (3.1)$$

where "plaq." is a primitive two-dimensional square (plaquette) on the d-dimensional hyper-cubic lattice, $R_1 R_2 R_3 R_4$ is the product of link variables R around the plaquette and the range of each link variable R is the Q roots of unity. The partition function for this model is written as

$$Z^{gauge} = \sum_{spin} e^{-S_g} \qquad (3.2)$$

Since $\frac{1}{Q} \sum_{n=1}^{Q} \left(R_1 R_2 R_3 R_4 \right)^n = \delta_{R_1 R_2 R_3 R_4 , 1}$, for Z_G^{gauge} (3.2) we have

$$Z_G^{gauge} = \sum_{spin} \prod_{plaq.} \left(1 + \tilde{v}\, \delta_{\Pi R_\ell , 1} \right) \qquad (3.3)$$

where $\tilde{v} = e^{K} - 1$.

Next, multiply out the product and represent the terms in the product by the subgraphs $G_f \subset G$ whose face sets correspond to the $\tilde{v}$ factors in the terms. Let $f(G_f)$ be the number of faces (plaquetts) in G_f, F - the number of faces in G, and $R_2(G_f)$ - the second Betti number [10] of the 2-dimensional manifold consisting of the G_f faces. The partition function then takes the following simple form after carrying out the spin summations

$$Z_G^{gauge} = \sum_{G_f \subseteq G} v^{f(G_f)} Q^{F - f(G_f) + R_2(G_f)} \tag{3.4}$$

T The expression (3.4) will be the starting point of the 1/Q-expansion for the gauge lattice model; it also shows explicitly how the topological invariants (i.e. Betti numbers) are connected with 1/Q-expansion. Moreover, this representation (3.4) for the partition function is useful for making different dual transformations.

It is known that three-dimensional gauge lattice model is dual to the spin one [12]. This fact can be easily proved using graph theory language for partition functions (2.4) and (3.4).

The theory of homologies states that if we have d-1-dimensional closed manifold P_{d-1} in d-dimensional space, then the space is divided into $R_{d-1}(P)+1$ -connected pieces, where $R_{d-1}(P)$ is (d-1)th Betti number of the manifold P [9]. Hence $n(G') = R_2(G_f) + 1$ where subgraph G' is dual to G_f. Now the duality transformation can be done in the same way as for the two-dimensional spin model (2.8)

$$Z_G^{spin}(v, Q) = \sum_{G'} v^{\ell(G')} Q^{n(G')} = \sum_{G_f} v^{F - f(G_f)} Q^{R_2(G_f)} = \tag{3.5}$$

$$= \left(\frac{v}{Q}\right)^{F} \sum_{G_f} \left(\frac{Q}{v}\right)^{f(G_f)} Q^{R_2(G_f) - f(G_f) + F} =$$

$$= \left(\frac{v}{Q}\right)^{F} Z_G^{gauge} \left(Q;\ \tilde{v}\right)$$

where $\tilde{v} = \dfrac{Q}{v}$, and the fact that $F = E = f\left(G^{f}\right) + \ell\left(G'\right)$ is used.

The analogous duality relations can be derived in any dimension d. The details will be published elsewhere.

4. Three-Dimensional Potts Models

Three-dimensional Potts models were investigated in Hamiltonian [11] and Lagrangian [3,12] formulation. According to [3] the expansion parameter depends on dimension, being in this case $1/Q^{1/3}$ for the Lagrangian formulation. But using (2.4) and (3.4) we can make 1/Q- expansion independently of dimension with the same parameter as it was in the Hamiltonian approach.

1/Q-expansion for spin and gauge models is efficient in the high-temperature phase. According to (3.4) we have the following 1/Q-expansion for the free energy per link

$$\begin{aligned} -K f_{h.t}^{gauge} &= 3\ln Q + \frac{3\tilde{v}}{Q} - \frac{3}{2}\frac{\tilde{v}^2}{Q^2} + \frac{\tilde{v}^3}{Q^3} - \frac{3}{4}\cdot\frac{\tilde{v}^4}{Q^4} + \\ &+ \frac{\tilde{v}^5}{Q^5}\left(\frac{3}{5} + \tilde{v}\right) - \frac{\tilde{v}^6}{Q^6}\left(\frac{3}{2} + 6\tilde{v}\right) + \frac{\tilde{v}^7}{Q^7}\left(\frac{45}{7} + 21\tilde{v}\right) - \\ &- \frac{\tilde{v}^8}{Q^8}\left(\frac{171}{8} + 56\tilde{v}\right) + \frac{\tilde{v}^9}{Q^9}\left(\frac{169}{3} + 129\tilde{v} + 3\tilde{v}^2\right) + \dots \end{aligned} \qquad (4.1)$$

In Fig.1 we've given an example of subgraph contributing to the 15-th order term in the 1/Q-expansion.

For the spin model from (2.4) we have

$$-\beta f^{spin}_{h.t} = \ln Q + \frac{3v}{Q} - \frac{3}{2}\frac{v^2}{Q^2} + \frac{v^3}{Q^3}(1+3v) - \\ - \frac{v^4}{Q^4}\left(\frac{15}{4} + 12v\right) + \frac{v^5}{Q^5}\left(18v^2 + 52v + \ldots\right) + \ldots \tag{4.2}$$

As in two dimension, we could calculate the free energy per site (link) due to the fact that only terms proportional to N (number links or sites) survive in the free energy 1/Q-ezpansion.

Since the three-dimensional spin model is dual to the gauge one (3.5), its low-temperature phase is the same as high-temperature one of the gauge model. Then for the low_temperature phase of the spin model we can write

$$-\beta f^{spin}_{\ell.t.} = 3\ln\frac{v}{Q} - K f^{gauge}_{h.t.}\left(\tilde{v} = \frac{Q}{v}\right) \tag{4.3}$$

where $\tilde{v} = e^{K} - 1$.

Shifting (4.2) and (4.3) we can receive a critical point

$$\tilde{v}_c = Q\left(1 + z + \frac{1}{3}z^3 + \frac{2}{3}z^4 - z^5 + \frac{20}{9}z^6 - \frac{4}{9}z^7 - 8z^8 + \right. \\ \left. + \frac{1985}{81}z^9 + \ldots\right) \tag{4.4}$$

This result coincides with that of Kogut [12] though the expansion parameter for the free energy was chosen in a different way.

Now the latent heat can be calculated by subtracting from each other the temperature derivatives of the free energy in the critical point.

Thus, using the graph theory language for partition functions of the Potts models, it is possible to make 1/Q-expansions for the models in any dimension d with the same expansion parameter.

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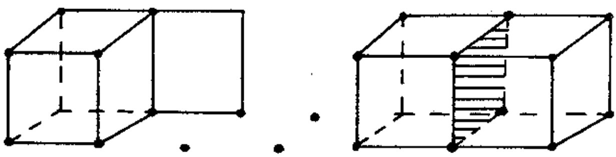
Fig.1. Subgraph contributing to the 15-th order in 1/Q-expansion with $R_2=2$, $f=17$ ($R_2-f = -15$).



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