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TWO-DIMENSIONAL ISING MODEL AND A SUPERSYMMETRIC QUANTUM PARTICLE

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ДВУХМЕРНАЯ МОДЕЛЬ ИЗИНГА И СУПЕРСИММЕТРИЧНАЯ КВАНТОВАЯ ЧАСТИЦА

Показано, что квантовая суперчастица с действием, предложенным в [1] ,в двухмерном евклидовом пространстве соответствует одному бозону и одной майорановской частице. Свободная энергия суперчастицы распадается на сумму свободных энергий бозона и майорановского фермиона, причем последняя совпадает со свободной энергией двухмерной модели Изинга вблизи критической точки.

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The quantum superparticle with an action proposed in [1] is shown to correspond in a two-dimensional Euclidean space to one boson and one Majorana particle. The superparticle free energy decomposes into the sum of free energies of a boson and a Majorana fermion, the latter coinciding with the free energy of the two-dimensional Ising model near the critical point.

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Introduction

A number of properties of the three-dimensional Ising model (which is dual to three-dimensional gauge model) indicates that near the critical point this system may be reduced to the fermionic string [2] by an analogy with the case of the two-dimensional Ising model, which, as is well known, is described near the critical point by free two-dimensional fermions. However, up to now the explicit method of the transition to the continuum limit of the three-dimensional Ising model is unknown, though a substantial progress is made in this direction [3]. Therefore, the search for such a variant of fermionic recording of the two-dimensional Ising model, whose immediate generalization for the string case could have described the near-critical behavior of the three-dimensional Ising model, becomes important.

In the present paper is considered the quantum superparticle with an action, invariant with respect to reparametrizations and transformations of the Poincare supergroup, first suggested in [1] (see also [4]). This supersymmetric theory is shown to include in the two-dimensional Euclidean space a boson and a fermion; the fermionic component of the continual integral over closed trajectories in the superspace coincides with the free energy of the two-dimensional Ising model near the critical point. The generalization of exactly such an approach for the string case will help to construct a continuum limit for the three-dimensional Ising model.

1. Supersymmetric Quantum Particle in Two-Dimensional Space

Consider the superspace with the coordinates $Z^M=(X_\mu, \theta)$. Here X_μ (μ = 1,2) are the coordinates of the two-dimensional Euclidean space, and θ are Grassmann variables realizing the bispinor representation of SO(2) group. An analogue to the Poincare group in this space is its minimum spinor expansion including, besides ordinary rotations and translations, also the translation along the spinor coordinates [5]

$$\begin{aligned} \theta &= \theta + \zeta \\ x'_\mu &= x_\mu + i\bar{\theta}\gamma_\mu\zeta, \end{aligned} \qquad (1)$$

where ζ is the anticommuting spinor parameter, and γ_μ are the Dirac matrices.

The form

$$ds^2 = (dx_\mu + i\bar{\theta}\gamma_\mu d\theta)^2, \qquad (2)$$

invariant with respect to the Poincare supergroup, is the generalized concept of the interval between the two points.

Consider the motion in the superspace of the relativistic free particle. Let $Z^M(\tau) = (x\ (\tau),\ \theta\ (\tau)\ )$ be the corresponding world line. By analogy with the ordinary expression for the action of particle as an integral of the length element, let's write down the action for the superparticle

$$S = -m_0 \int ds = -m_0 \int_0^T d\tau \sqrt{(\dot{x}_\mu + i\bar{\theta}\gamma_\mu \dot{\theta})^2}\ , \qquad (3)$$

where

$$\dot{x}_\mu = \frac{dx_\mu}{d\tau}\ ,$$

$$\dot{\theta} = \frac{d\theta}{d\tau}\ ,$$

m_0 is the bare mass. The action (3) is invariant with respect to the Poincare group and group of reparametrizations $\tau \to \tau'(\tau)$.

Let's now turn to the corresponding quantum theory. Following the Feynman procedure of quantization, the particle propagator is written in the form of a path integral

$$K = \int \mathcal{D}x_\mu\, \mathcal{D}\theta \exp\left\{-m_0 \int_0^T d\tau \sqrt{(\dot{x}_\mu + \frac{i}{m_0}\bar{\theta}\gamma_\mu \dot{\theta})^2}\right\}. \qquad (4)$$

In (4) we pass on to dimensionless θ by the formula $\theta \to \frac{1}{\sqrt{m_0}}\theta$. Such a quantum superparticle contains one scalar particle and one Majorana fermion. In order to see this, consider the free scalar superfield

$$\Phi(x,\theta) = A(x) + \theta\psi(x)$$

with the action

$$S_1 = \int d^2x\, d^2\theta\, \phi^+(x,\theta)(\bar{D}_\alpha D_\alpha - m_0^2)\phi(x,\theta) \qquad (5)$$

where

$$D_{\alpha}=\left(\frac{\partial}{\partial\theta_{\alpha}}+\frac{1}{4}(\bar{\theta}\hat{\partial})_{\alpha}\;,\partial_4\right),$$
$$\bar{D}_{\alpha}=\left(-\frac{\partial}{\partial\bar{\theta}_{\alpha}}+\frac{1}{4}(\hat{\partial}\theta)_{\alpha}\,,\partial_4\right) \qquad (6)$$

are the covariant derivatives. The generating functional of this theory is

$$Z=\int\mathcal{D}\phi e^{-S_1}=\left[\det(\bar{D}D-m_0^2)\right]^{1/2}=e^{-F} \qquad (7)$$

$$F=\frac{1}{2}\operatorname{Tz}\ln(\bar{D}D-m_0^2). \qquad (8)$$

Using the Schwinger proper time formalism, it is easy to show that

$$F=\int\mathcal{D}x\,\mathcal{D}\theta e^{-S}, \qquad (9)$$

where the integration is carried out over all closed trajectories with a nonfixed initial point, and S is defined by the expression (3). Thus the theory of quantum superparticle is equivalent to that of scalar superfield containing a boson and a fermion. In the next section this fact will be considered from a different viewpoint.

2. Two-Dimensional Ising Model Near the Critical Point and Quantum Superparticle

It is known [6], that the statistical sum of the two-dimensional Ising model may be presented as a sum over non selfintersecting trajectories on a regular lattice. The elimination of self-intersecting trajectories is attained by ascribing a sign factor $(-1)^n$ to each trajectory, where n is the number of self-intersections. Thus the statistical sum reads

$$Z = \sum \lambda^L (-1)^n \qquad (10)$$

Near the critical point there is a total O(2) symmetry, which is expressed in the fact, that one may sum in (10) not only over regular trajectories, but also over arbitrary broken ones. This is equivalent to renormalization of the quantity λ. It is only necessary that all the trajectories are taken into account with the factor $(-1)^n$. Then the free energy may be written as [7]

$$F_1 = -\int \mathcal{D}x\, e^{-mL}\, \mathrm{Tr}\, P\exp\left\{\frac{i}{2}\gamma_5 \oint d\phi\right\}, \qquad (11)$$

where $d\phi$ is the element of the turn angle of tangential vector to the loop. Since, on the other hand, the two-dimensional Ising model near the critical point is equivalent to the free Majorana particle theory, F_1 is the free energy of the Majorana fermion with a bare mass m. Finally, since the Majorana fermion is one of the components of quantum superparticle, it is quite natural to expect that F_1 from (11) will be included in free energy F(9) as a component. We shall now show that this is the very case:

$$F = F_0 + F_1 \qquad (12)$$

where F is given by (9), F_1 by (11), and F_0 is the free energy of bosonic particle.

As a proof we calculate in (9) the integral in θ (τ). To do this let us expand the integrand in terms of θ.

$$-m_0\sqrt{(\dot{x}_\mu+\frac{i}{m_0}\bar{\theta}\gamma_\mu\dot{\theta})^2}=-m_0\sqrt{\dot{x}^2}+i\bar{\theta}\gamma_\mu e_\mu\dot{\theta} \tag{13}$$

$$-\frac{1}{2m_0\sqrt{\dot{x}^2}}(\bar{\theta}\gamma_\mu\dot{\theta})(\bar{\theta}\gamma_\nu\dot{\theta})(\delta_{\mu\nu}-e_\mu e_\nu),$$

where $e_\mu=\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}$ is the unit tangential to the vector trajectory. Let's now discretize the space of the parameters $\tau\in[0,T]$ in such a way, that the quantity $(\Delta x_\mu)^2=(x_{\mu,n+1}-x_{\mu,n})^2=\varepsilon^2$ is constant along the trajectory. We make in (13) the following replacements

$$\begin{aligned} e_\mu(\tau) &\longrightarrow \frac{e_{\mu,n}+e_{\mu,n+1}}{2},\\ \theta(\tau) &\longrightarrow \frac{\theta_n+\theta_{n+1}}{2},\\ \dot{\theta}(\tau) &\longrightarrow \frac{\theta_{n+1}-\theta_n}{\varepsilon}. \end{aligned} \tag{14}$$

Let's write the measure of the integration in $\theta(\tau)$ in the form of $\prod_n d^2\theta_n$, where $d^2\theta_n=-i\,d\theta_{1n}\,d\theta_{2n}$ The expression (9) looks like

$$F=\int\prod_n dx_{\mu,n}\,e^{-m_0L}\prod_n d^2\theta_n$$

$$\exp\sum_n\Big[i\bar{\theta}_n\frac{\hat{e}_n+\hat{e}_{n+1}}{2}\theta_{n+1}-\frac{1}{2m_0\varepsilon}(\bar{\theta}_n\gamma_\mu\theta_{n+1}) \tag{15}$$

$$(\bar{\theta}_n\gamma_\nu\theta_{n+1})\Big(\delta_{\mu\nu}-\frac{e_{\mu,n}+e_{\mu,n+1}}{2}\;\frac{e_{\nu,n}+e_{\nu,n+1}}{2}\Big)\Big]$$

where L is the loop length, $\hat{e}_n=e_{\mu,n}\gamma_\mu$.

Let us expand the exponent in (15) in Taylor series and use the relation

$$\theta^2_{\alpha n}=0, \qquad \int d^2\theta_n\, \theta_{\alpha n}\,\bar{\theta}_{\beta n}=i\delta_{\alpha\beta}, \tag{16}$$

i.e. consider only those terms of the expansion that contain a pair of $\theta_{\alpha n}\ \bar{\theta}_{\beta n}$ for each n. The first component in the exponent in (15) then gives

$$-Tr\prod_n \frac{\hat{e}_n+\hat{e}_{n+1}}{2} = -Tr\prod_n \frac{1+\hat{e}_n e_{n+1}}{2} \tag{17}$$

The trace of the matrix product has arisen here since we should consider the closed loops only. It is easy to see that

$$\hat{e}_n\hat{e}_{n+1} = e_n e_{n+1} + i\gamma_5\,|e_n\times e_{n+1}| =$$
$$= \cos\Delta\phi_n + i\gamma_5\sin\Delta\phi_n ,$$

where $\Delta\phi_n$ is the angle between the vectors e_n and e_{n+1}. Therefore, the expression (17) is equal to

$$-\prod_n \cos\frac{\Delta\phi_n}{2}\; Tr\, P e^{\frac{i}{2}\gamma_5\sum_n \Delta\phi_n} \tag{18}$$

The contribution of the fourth order terms in θ into the exponent expansion in (15) is apparently

$$\prod_n\left[\left(\frac{1}{m_0\varepsilon}-1\right)\left(\frac{\hat{e}_n+\hat{e}_{n+1}}{2}\right)^2-\frac{2}{m_0\varepsilon}\right]=$$
$$=\prod_n\left[\left(\frac{1}{m_0\varepsilon}-1\right)\cos^2\frac{\Delta\phi_n}{2}-\frac{2}{m_0\varepsilon}\right]. \tag{19}$$

Joining the expressions (18) and (19) we obtain

$$
\begin{aligned}
F &= \int \mathcal{D}x_\mu e^{-m_0 L}\left[\frac{2}{m_0\varepsilon}-\left(\frac{1}{m_0\varepsilon}-1\right)\cos^2\frac{\Delta\phi_n}{2}\right]- \\
&- \int \mathcal{D}x_\mu e^{-m_0 L}\prod_n \cos\frac{\Delta\phi_n}{2}\,\mathrm{Tr}\,P e^{\frac{i}{2}\gamma_5\sum_n \Delta\phi_n} \\
&= \int \mathcal{D}x_\mu e^{-mL} - \int \mathcal{D}x_\mu e^{-m_1 L}\,\mathrm{Tr}\,P e^{\frac{i}{2}\gamma_5\sum_n \Delta\phi_n} \\
&= F_0 + F_1
\end{aligned}
\qquad (20)
$$

where

$$
m = m_0 - \frac{1}{\varepsilon}\ln\left[\frac{2}{m_0\varepsilon}-\cos^2\frac{\langle\Delta\phi\rangle}{2}\left(\frac{1}{m_0\varepsilon}-1\right)\right],
$$

$$
m_1 = m_0 - \frac{1}{\varepsilon}\ln\cos\frac{\langle\Delta\phi\rangle}{2}
$$

are the renormalized masses, $\langle\Delta\phi\rangle$ is the particle deviation mean angle at random walk.

Thus, we see that the free energy of superparticle decomposes into bosonic (F_0) and fermionic (F_1) parts, with F_1 coinciding with the free energy of the two-dimensional Ising model near the critical point.

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