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S.G. MATINYAN

# DYNAMICAL CHAOS OF NON-ABELIAN GAUGE FIELDS

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# DYNAMICAL CHAOS OF NON-ABELIAN GAUGE FIELDS*

A review on proof and investigation of the stochasticity of the classical Yang-Mills fields (the classical Yang-Mills mechanics) is given. Also the classical Yang-Mills-Higgs system is considered. Some features of quantum Yang-Mills mechanics are described. It is shown that if the contribution to the functional integral of theory is given only by fields generated by random currents, then the confinement of the corresponding fields takes place.

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С.Г.МАТИНЯН

# ДИНАМИЧЕСКИЙ ХАОС НЕАБЕЛЕВЫХ КАЛИБРОВОЧНЫХ ПОЛЕЙ*

Дан обзор по доказательству и исследованию стохастичности классических полей Янга-Миллса (классическая механика Янга-Миллса). Рассмотрена также классическая система Янга-Миллса-Хиггса. Описаны некоторые черты квантовой механики Янга-Миллса. Показано, что если в функциональный интеграл теории дают вклад лишь поля, генерированные случайными токами, то имеет место конфайнмент соответствующих полей.

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## 1. Introduction

Until recently, a random (chaotic) behaviour of dynamical systems was associated with either random initial conditions or action of random external forces (as, e.g., in the case of Brownian motion), or, finally, with excitation of a very large number of degrees of freedom. Of course, any of these conditions is enough to give rise to chaos in systems; however, as it became known, none of them may be treated as necessary condition [1-6].

At present, it is well established that a large number of simple completely determined dynamical systems of classical mechanics with a small number ( $n \geqslant 2$) of degrees of freedom is characterized by extremely irregular, exceptionally complicated and practically unpredictable motion determined entirely by intrinsic dynamics of a system.

Such a random (stochastic) behaviour, nowise connected with the above-quoted sufficient conditions of occurrence of chaos, is natural to determine as dynamical stochasticity.

A mechanism of occurrence of such dynamical chaos consists in strong local instability of motion [3, 4, 7]. Typical random features manifest themselves even in a separate trajectory of a system.

A dynamical chaos is typical of many nonlinear classical systems in various fields of physics, and also other sciences (chemistry, hydrodynamics, biology, meteorology, ecology, etc.).

We shall see below that it manifests itself specifically also in classical theory of non-Abelian gauge fields.

The question of complete integrability (or, more precisely, nonintegrability) of the classical Yang-Mills (Y.M.) equations, associated directly with the problem of their stochasticity, has already its history and is to a certain extent connected with wide popularity of classical solutions, e.g., of the instanton type [8, 9], on which great hopes were set for the construction of QCD ground state.

However all attempts to find additional integrals of motion of the classical Y.M. equations have hitherto been unsuccessful. As a result, a program of searches for conservation laws has arisen, being expressed not in terms of potentials and fields, but in terms of the manifold of loops [10, 11].

This circumstance stimulated investigations of the classical Y.M. equations, nonlinear in their nature, as concerning presence of stochastic component in the latter. The urgency of studying non-Abelian gauge fields from the viewpoint of stochasticity, besides being of great interest itself, is dictated also by a phenomenon (originating from solid state physics) of dimensional reduction in quantum spin systems interacting with random magnetic field [12]. In 1982, P.Olesen [13] suggested a hypothesis that by analogy with this phenomenon, random fields reduce the 4-dimensional Y.M. theory to the effective two-dimensional one which possesses the confinement property. He showed that in the limit of infinitely large number of colors, $N \to \infty$ , the presence of random fields in the vacuum is a necessary and sufficient condition of confinement. In Ref. [14] , on the example of calculation of Wilson average $W(C)$ in the limit $N \to \infty$ and confined by planar loops, this reduction is observed concretely. Calculations in the SU(2) lattice gauge theory [15] also point out a reliability of Olesen's hypothesis.

The quoted considerations show that the problem of confinement may be solved if one is convinced that random vacuum fields naturally arise in the 4-dimensional QCD, being its essential part.

The present review deals with works in which the stochasticity of sourceless classical non-Abelian gauge fields is observed and proved [16-18]

So, finally one may conclude that QCD, as distinct from quantum electrodynamics, is a theory which in classical limit has strongly developed stochastic features.

What happens with stochasticity when we proceed to quantum systems seems a rather complicated problem being far from its final solution.

In general [19, 20] , it should be expected that dynamical stochasticity cannot take place in quantum systems with limited phase space, because the wave function (or the density matrix) of such systems is always quasi-periodical, i.e. its spectrum is discrete. A transient or temporary stochasticity may, at the best, take place in such systems. One can say that until, in quasi-classical terms, wave packet of such a system diffuses, the latter will have a classical, and hence, stochastic trajectory, and then at least the stronger stochastic features must vanish.

However in the quantum case for conservative systems, one should scarcely speak about trajectories, even in the quasi-classical limit. The concepts of spectrum and wave functions of the system are more appropriate here. What are the properties of spectrum of quantum system which in classical limit exhibits stochastic motion? This question is quite natural and highly important.

It seems reasonable enough [21] that in the quasi-classical limit the quantum energy spectrum of a dynamical system consists of a regular and irregular parts. In the general case, a regular part of the spectrum (weakly varying with Hamiltonian parameters) corresponds at $\hbar = 0$ to regular classical trajectories which are winding of invariant torus.

The irregular part of the spectrum (strongly dependent on Hamiltonian parameters) in this limit corresponds to dynamical chaos.

A numerical simulation shows [22, 23] that there is a correspondence between a fraction of classical chaotic motion and that of irregular part of a manifold of the energy eigenvalues. Such part of spectrum arises beyond the critical energy, at which a classical regular motion starts to turn into chaotic one.

Note that the irregular part of spectrum, taking into account the Hamiltonian symmetry properties, is, as a rule, connected with repulsive levels in accordance with the well known theorems [24, 25].

We will return once again (in Section 5) to the question on nature of spectrum of quantum systems which in classical limit have stochastic component.

Another criterion that discriminates between regular and irregular (i.e. corresponding to chaos in classical limit) quantum states is connected with the behaviour of wave functions: to the first case a regular interference pattern and large intensity fluctuations correspond, while to the second case randomly distributed interference maxima and minima with suppressed intensity fluctuations do.

Things are different when the quantum system is unclosed, i.e. when it is in chaotic external field. The above-stated arguments, generally speaking, are inapplicable in this case, so the question needs a special study. Recently carried out consideration of such simple systems [27] shows that the quantum-mechanical properties of these systems, stochastic in the classical limit, do not impose strong limitations on the stochasticity manifestations. In other words, the considered quantum systems in random field exhibit continuous spectral properties, just as the corresponding classical model does.

The above-quoted considerations make it plausible that the discovered dynamical stochasticity of the free classical non-Abelian gauge fields will leave its traces in a real world of QCD, so we may hope that just these phenomena are responsible for the color confinement.

## 2. Space-Homogeneous Yang-Mills Fields. Exact Solution. Classical Yang-Mills Mechanics.

At present, there are many reasons to state that the perturbational vacuum of Yang-Mills theory does not coincide with the true one. The arguments in favour of this statement have both classical [8, 9] and quantum [28, 29] basis.

Qualitatively, owing to gluons interaction ("pairing"), their condensate arises, manifesting itself in nonzero vacuum expectation value of squared field tensor of gluons and lowering the energy of ground state of QCD which ignores this phenomenon.

From the classical point of view, the search and analysis of the classical solutions of Y.M. sourceless equations in Minkowski space, that could serve as a basis for constructing and studying the QCD vacuum structure and the asymptotic states problem, seem highly important.

Let us start our consideration with the free Y.M. fields in ordinary space - time corresponding to the SU(2) group.

The equations of motion have the form

$$\partial_\mu G^a_{\mu\nu} + g\varepsilon^{abc} A^b_\mu G^c_{\mu\nu} = 0 \tag{2.1}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu$$

( here and below the Latin indices range the values 1, 2, 3, the Greek ones- 0, 1, 2, 3).

We shall look for a class of solutions of the system (2.1), for which the Poynting vector in some system vanishes [30] :

$$T_{0j} = G^a_{0i} G^a_{ji} = 0 \tag{2.2}$$

( $T_{\mu\nu} = - G^a_{\mu\lambda} G^a_{\nu\lambda} + \frac{1}{4} g_{\mu\nu} G^{a\,2}_{\lambda\rho}$ is the energy-momentum tensor of the field).

In the gauge $A^a_0 = 0$ Eqs.(2.1) and condition (2.2) take the form

$$\ddot{A}^a_i - G^a_{ji,j} + g\varepsilon^{abc} A^b_j G^c_{ji} = 0 \tag{2.1a}$$

$$N^a \equiv \varepsilon^{abc} A^b_i \dot{A}^c_i = 0 \tag{2.1b}$$

$$\dot{A}^a_i G^a_{ij} = 0 \tag{2.2a}$$

(the dot over $\dot{A}^a_i$ denotes the time differentiation, $G_{ij,k} \equiv \partial_k G_{ij}$ ), where Eq.(2.1b) plays a role of constraint. $N^a$ vanishes for the free equations. In case of the presence of external current, $- N^a$ is the density of external color charge $j_0^a$ .

The constraint equations (2.1b) and (2.2a) lead to a relation

$$\dot{A}_i^a \left( A_{j,i}^a - A_{i,j}^a \right) = 0 . \tag{2.2b}$$

A sufficient condition of the validity of this relation are

a) $A_{i,j}^a = 0$, b) $\dot{A}_i^a = 0$, c) $A_{j,i}^a - A_{i,j}^a = 0$.

We shall examine, as it will be seen in the following, a most interesting case a) of space-homogeneous Y.M. fields, when in the given coordinate system the fields depend on time only

$$A_i^a = A_i^a (t).$$

The region of applicability of equations for homogeneous fields is determined by a condition that time variations dominate in the system. In other words, this corresponds to a long-wave part of the spectrum (or to strong fields):

$$| A_{i,j}^a | << A_i^{a\,2} , \quad \lambda | A_i^a | >> 1 .$$

We may hope that the study of such fields will be helpful for obtaining information on the QCD infrared regime - its most unsolved point.

The equations of motion for homogeneous fields will take the form:

$$\ddot{A}_i^a - g^2 A_j^a A_j^b A_i^b + g^2 A_i^a A_j^b A_j^b = 0 \tag{2.3}$$

with constraint ([illegible]).

Thus, for space-homogeneous Y.M. fields the field equation (2.1) reduces to a discrete nonlinear mechanical system with a Hamiltonian

$$H_{YM} = \sum_{i,a=1}^{3} \frac{1}{2}\left(\dot{A}_i^a\right)^2 + \frac{g^2}{4}\left[\left(A_i^a A_i^a\right)^2 - \left(A_i^a A_j^a\right)^2\right] \tag{2.4}$$

One can readily see that this Hamiltonian is symmetric relative to the matrix $A_i^a$ transposition, i.e. relative to internal and "external" (three-dimensional) spaces, both isotropic ( the O(3) x O(3) symmetry ), therefore, as we can easily see, two "moments" are conserved: a usual three-dimensional moment

$$M_i = \varepsilon_{ijk} A_j^a \dot{A}_k^a$$

and internal "three-dimensional moment" $N^a$ (2.1b) which is nonzero only for fields with sources and equals to $-\dot{j}_o^a$ .

The above-said provides one with many reasons as to call the considered system of homogeneous Y.M. fields the classical Yang-Mills mechanics, which, as will be shown below, exhibits dynamical stochasticity in full measure.

The system (2.3) has nine degrees of freedom ( $i$, $a$ = 1, 2, 3) and four trivial conserved integrals $H_{YM}$ and $M_i$ .

Before proceeding to the analysis of a situation with the number of degrees of freedom $n \geq 2$, let us consider a simple case [30]. We shall seek a solution to the system (2.3) in the form

$$A_i^a = \frac{O_i^a}{g} f^{(a)}(t) \tag{2.5}$$

(there is no summation over $a$ in (2.5) ), where $O_i^a$ is a constant orthogonal matrix

$$O_i^a \; O_i^b = \delta^{ab}. \tag{2.6}$$

For $f^{(a)}(t)$ we obtain the following system

$$\ddot{f}^{(a)} + f^{(a)}\left(\vec{f}^{\,2} - f^{(a)2}\right) = 0 \qquad (2.7)$$

where

$$\vec{f}^{\,2} = \sum_{a=1}^{3} f^{(a)2} .$$

It is known that any conservative systems with one degree of freedom are integrable, so a particular solution of the system (2.7) at $f^{(1)} = f^{(2)} = f^{(3)} = f(t)$ can readily be found using the energy integral

$$f(t) = \left(\frac{2g^2}{3}\right)^{1/4} \mu\, \mathrm{cn}\left[\left(\frac{8g^2}{3}\right)^{1/4} \mu t\, ;\ 1/\sqrt{2}\,\right] \qquad (2.8)$$

where $\mathrm{cn}(x;\,k)$ is the Jacobi elliptic cosine of argument $x$ and modulus $k$, $\mu^4$ is the Hamiltonian density $T_{oo}$ in the considered coordinate system.

The solution given by formulae (2.5), (2.6), (2.8) varies in time with period

$$T = \left(\frac{3}{8g^2}\right)^{1/4} \frac{4}{\mu}\ K\left(1/\sqrt{2}\,\right),$$

where $K(x)$ is a complete elliptic integral of the first kind.

Note some interesting features of the solution obtained, though they do not relate directly to the question of stochasticity of Y.M. equations we are interested in.

The corresponding to this solution field strengths

$$E_i^a = \frac{O_i^a}{g}\,\dot{f}\,, \qquad H_i^a = \varepsilon_{ijk}\,\varepsilon^{abc}\,\frac{O_j^b\,O_k^c}{2g}\,f^2 .$$

$$\left(H_i^a = g\left(\frac{f}{\dot{f}}\right)^2 \varepsilon_{ijk}\,\varepsilon^{abc}\,E_j^b\,E_k^c\right)$$

are such that $\vec{E}^a(\vec{H}^a)$ are mutually orthogonal in the "rest" frame, and $\vec{H}^a$ are parallel to $\vec{E}^a$ ( $a = 1, 2, 3$).

Further on, one can easily see that the argument of the periodical solution (2.8) in an arbitrary frame obtained from our system for the "accompanying" wave, will transform under the corresponding Lorentz boost into $\xi \equiv \kappa x = \kappa_\mu x_\mu$, where $K_0 = \mu\gamma$ , $K_i = \mu\gamma v_i$ , ( $\gamma = (1 - v^2)^{-1/2}$ , i.e. $K^2 = \mu^2$ . Such a solution cannot take place in linear massless electrodynamics since it is impossible to choose a coordinate frame in which the magnitude of the Poynting vector of a wave is equal to zero and not to the energy density. Just this is responsible for the difference between the solution (2.8) and the corresponding Coleman's solution [31] . Owing to the same circumstance, $\mu$ formally plays a role of mass in nonlinear wave (2.8).

Of course, we could from the start seek a solution of the system (2.3) in the form $A_i^a(x) = A_i^a(\xi)$ with $K^2 = \mu^2$ . However in this case, the analogy with classical dynamical system to which we have reduced gauge field would be less explicit.

## 3. Two Degrees of Freedom. Qualitative Analysis of Color Oscillations [16] .

Nonlinear system with $n = 2$ in the conservative case already can possess all characteristic features of dynamical stochasticity.

For the corresponding Hamiltonian system (2.3), introducing notations $A_1^1 = \frac{1}{g} x(t)$ , $A_2^2 = \frac{1}{g} y(t)$ and taking $A_2^1 = A_1^2 = 0$ we arrive at a nonlinear mechanical system on plane with Hamiltonian

$$H = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{x^2 y^2}{2} \tag{3.1}$$

and the corresponding very symmetric and simple in form constraint equations of motion which we shall investigate here:

$$\begin{aligned} \ddot{x} + x y^2 = 0 \\ \ddot{y} + y x^2 = 0 \end{aligned} \tag{3.2}$$

The analysis of these equations is undoubtedly much simpler than that of a more general system (2.3) and all the more, of (2.1). However, if the stochasticity of the system (3.2) will be shown, it can hardly be conceived that a stochastic component would disappear entirely in a more complicated system with $n > 2$ and all the more, in the general case of space-inhomogeneous Y.M. fields. In the following sections we shall consider homogeneous Y.M. fields with $n > 2$.

It follows from the form of $H$ (3.1) that any conserved integral $F(x, y, \dot{x}, \dot{y})$ of the system (3.2) must satisfy the partial differential equation

$$\dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} = xy\left[y\frac{\partial F}{\partial \dot{x}} + x\frac{\partial F}{\partial \dot{y}}\right]$$

from which it can be seen that $F$ cannot depend on only two of variables $x, y, \dot{x}, \dot{y}$ or be a polynomial of finite degree in these variables.

It is obvious that the "material point" described by (3.2) cannot leave the region bounded by the equipotential curves $xy = \pm\sqrt{2}\,\mu^2$, where $\mu^4$ is the "total energy of the point". It is obvious that if the "point" with "total energy" $\mu^4$ described by (3.2) is at some instant on the equipotential curve $xy = \pm\sqrt{2}\,\mu^2$, then it will leave this curve along the normal into the allowed region.

Let us see whether the system (3.2) has periodic trajectories.

It follows from the symmetry of the problem that the trajectory will be periodic if any of the events listed below occur at least twice: a) the trajectory passes through the origin; b) the trajectory is perpendicular to one of the symmetry axes; c) the trajectory reaches the equipotential curve.

These sufficient conditions of periodicity are helpful for the classification and description of the trajectories (given below), however we do not rule out that one could find other weaker sufficient criteria of periodicity of the trajectories of the system (3.2).

Along the symmetry axes $x = \pm y$ the system executes, of course, the periodic oscillations (2.8) (events a) and c) ). Along the axes $= 0$ and $y = 0$ the point, as in electrodynamics, goes away to infinity ($\ddot{x} = 0$, $\dot{x} \neq 0$; $\ddot{y} = 0$, $\dot{y} \neq 0$). But if at some instant the velocity of the point is not directed along the $x$ or $y$ axis, then it will not go to infinity, though in some cases it may travel an arbitrarily large distance from the centre and return in a finite time to the region $x \sim y$, as is readily seen from the negativity of $\ddot{x}/x$ and $\ddot{y}/y$.

One can say that such a motion occupies an intermediate position between finite and infinite motions.

In polar coordinates ( $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ ) Eqs.(3.2) have the form

$$\ddot{\varphi} + \frac{2\dot{\rho}}{\rho}\dot{\varphi} + \frac{\rho^2}{4}\sin 4\varphi = 0 \qquad (3.3)$$

$$\ddot{\rho} - \rho\dot{\varphi}^2 + \frac{\rho^3}{2}\sin^2 2\varphi = 0. \qquad (3.3")$$

In the case of motion away from the centre ( $\rho \gg \mu$ ; note that in our problem $x$, $y$, $\rho$ have dimensions of mass and not length! ), for example, along the channel $\varphi \ll \pi/4$ , $\sin 4\varphi \approx 4\varphi$ , $\dot{\rho} > 0$, it can be seen from (3.3 ) that the frequency of the oscillations with respect to the coordinate $\varphi$ increases with increasing distance from the centre, while the amplitude decreases until $\dot{\rho} = 0$ ("turning point") (this last occurs in a finite interval of time, since $\ddot{\rho} \simeq a(t)\rho^3(t)$ ( $a > 0$) ), after which the "damping" regime is replaced by a "swinging" regime. Figure 1 shows a characteristic example of such behaviour obtained on a computer.

The motion with respect to $\rho$ , averaged over the rapid oscillations of $\varphi$ , consists of a random walk with large amplitudes ( $\ddot{\rho} + a\rho^3 \approx 0$ ) from channel to channel with complicated motion in the region $x \sim y$ (which can be followed in a numerical integration of the system (3.2) on a computer). In the language of the variation in time of the color amplitudes $A_1^1$ and $A_2^2$ , this picture corresponds alternately to rapid oscillations and decrease of one color amplitude and growth of the other.

It is obvious that the behaviour of the three-dimensional system is qualitatively similar to the behaviour of the system (3.2) with $n = 2$ (see below) that we considered above. In this case there are six channels along the coordinate axes and the motion in them is analogous to the motion in the channels of the two-dimensional system, i.e., with increasing distance from the centre, the frequency of the oscillations of the trajectories increases, and the amplitude decreases until it stops, after which the regime of damping with respect to the spherical angle is replaced by a swinging regime. The general picture of the variation in time of the color amplitudes in this three-dimensional case is characterized by alternate rapid oscillations and decrease of two color amplitudes and growth of the third. "Beats" of the color take place.

## 4. Instability of Periodic Trajectories of the System (3.2). Stochasticity.

In Figure 2 we show examples of some periodic trajectories photographed on the display of the computer used to integrate the system (3.2) [16] .

In Figs. 2(a)-2(f) we show trajectories that pass through the coordinate origin and are perpendicular to either an equipotential line (Figs.2(a) 2(b), 2(d), and 2(f) ) or the symmetry axis $y = 0$ (Figs. 2(c) and 2(e) ). The trajectories are arranged in the order of decreasing slope relative to the $x$ axis at the origin. The trajectory in Fig.(a) corresponds to the oscillations in accordance with the elliptic cosine law (2,8) [30] . A further decrease in the slope leads to an increase in the number of intersections with the $x$ axis as in the trajectories in Figs.2(c) and 2(e) and 2(d) and 2(f).

We denote these angles for trajectories of the type in Figs.2(c) and 2(e) by $\alpha_n^0$ and for trajectories of the type in Figs.2(d) and 2(f) by $\beta_n^0$ , where $n$ is the number of intersections of the trajectories with the $x$ axis.

In the limit $n \rightarrow \infty$ , the angles $\alpha_n^0$ and $\beta_n^0$ tend to zero.

These figures clearly reveal the tendency to an increase in the frequency and decrease in the amplitude of the oscillations as the particle moves further into the channel along the $x$ axis the smaller is the angle between the $x$ axis and the trajectory at the origin - in agreement with the qualitative analysis made above (Sec.3) for large $\rho$ .

In Figs.2(g)-2(m) we show examples of trajectories that pass perpendicular to the $y$ axis at different distances from the centre and perpendicular to either the coordinate axis (Figs.2(g), 2(j), 2(l), and 2(m)) or the equipotential lines. With decreasing distance of these trajectories along the $y$ axis from the centre, they all then enter the channel, and the picture considered in Sec.3 is again reproduced qualitatively. Figs.2(p) 2(q), and 2(r) show trajectories that are twice perpendicular to the equipotential lines.

Finally, Figs.2(s)-2(x) represent trajectories perpendicular to the symmetry axes $x = \pm y$ .

On the basis of the above analysis of the trajectories in Fig.2 it can be seen that the number of periodic trajectories of the type in Figs.2(c)-2(f), and also of the type in Figs.2(n) and 2(m) is countable, so that we can assert that the set of periodic solutions of the system (3.2) is at least countable.

Since no trajectory of the system (3.2) can lie entirely in a single quadrant of Fig.1, it follows from this and the symmetry of the problem that all possible trajectories of the system can be obtained by specifying initial conditions in the form

$$y = 0, \quad x = x_0 > 0, \quad \dot{x} = \sqrt{2}\,\mu^2 \cos\alpha, \quad \dot{y} = \sqrt{2}\,\mu^2 \sin\alpha \quad (0 \leq \alpha \leq \pi).$$

From this analysis, most important to us is the fact that, as we have seen, the trajectories of the system are extremely unstable with respect to small changes in the initial conditions ( $x_0$, $\alpha$ ), which is one of the signs of stochasticity of the system (3.2).

Following the arguments quoted above, one should expect that the trajectories of the system possess local instability which is just the reason of strong dependence of the motion on both initial conditions and different small perturbations. Another method, that shows clearly this stochasticity, is connected with the computer experiments applying Poincare's mapping method [32] . Experiments of such type were originally carried out when studying the stellar motion in the galaxy field (Henon, Heiles, Contopoulos, Ford et al. [33-35] ).

A computer was programmed to solve Eqs.(3.2), as well as to light out the points of intersection of phase trajectory of the system in the space ( $x, \dot{x}, y, \dot{y}$ ) with the plane ( $y, \dot{y}$ ) at $\dot{x} > 0$ [18]. If the motion is periodic, then the intersection occurs in a finite number of points; if the system is integrable, i.e. the trajectory represents tore winding, then the points construct a regular closed curve in the plane ( $y, \dot{y}$ ). If, finally, the behaviour of the system is stochastic, the point of intersection travels randomly in the plane ( $y, \dot{y}$ ) and covers densely the finite area. Precisely such behaviour of the trajectories of the system (3.2) in the plane ( $y, \dot{y}$ ) at $\dot{x} > 0$ is revealed by a computer [18] , which appears a proof of the stochasticity of the system (3.2) (see Fig.3, where all the points intersecting the plane ( $y, \dot{y}$ ) belong to the same trajectory). The most characteristic and important property of random motion is rapid exponential divergence of close phase trajectories: $R \sim e^{ht}$ , where $h > 0$ .

This criterion of dinamical chaos is especially helpful in numerical simulation.

The quantity $h$ which determines the exponential rate of divergence of close trajectories is so-called metric entropy of a chaotic component of motion, called sometimes KS-entropy (the Krylov-Kolmogorov-Sinai entropy).

If $h > 0$ , then the motion has a stochastic component. Moreover, the condition $h > 0$ , according to modern theory of dynamical systems [7] , is necessary and sufficient for stochasticity of nearly all trajectories [36] .

KS-entropy is determined by Lyapunov's exponents $\Lambda_i \ (> 0)$

$$h = \sum_{i=1}^{n-1} \Lambda_i \geqslant \Lambda_m \tag{4.1}$$

where $\Lambda_m$ is a maximal exponent ( $n$ is, in most cases, the number of degrees of freedom of the system).

$\Lambda_m$ is determined by a "distance" between close trajectories in the phase space

$$\rho^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta \dot{x}) + (\Delta \dot{y})^2 + (\Delta \dot{z})^2$$

(for definiteness, we consider the case with $n = 3$ , $A_1^1 = x$ , $A_2^2 = y$ , $A_3^3 = z$ that was studied originally with the use of the notion of KS-entropy in Ref. [17] ):

$$\Lambda_m = \lim_{t \to \infty} \frac{\ln \rho(t)}{t} . \tag{4.2}$$

The behaviour of close trajectories we study in the linear approximation which is quite correct since we are interested in a strictly local bahaviour of close trajectories.

If $\Lambda_m > 0$ , then it follows from (4.1) that $h > 0$ , and close trajectories diverge exponentially. For an integrable system (the quasi-periodic motion) $\rho(t) \sim t$ (the power local instability) and $\Lambda_m = h = 0$

In Ref. [17] , making use of the notion of KS-entropy, the exponential local instability of the system of the type of (3.2) with $n = 3$ was shown, as well as the above conclusion [16] on the stochasticity of the system (3.2) with $n = 2$ was confirmed.

A more detailed analysis of Eqs.(3.2) from the viewpoint of the presence of the stochastic component in them, can be found in Ref. [36], where the chaotic component is shown to cover in both cases nearly the whole energy surface.

One more criterion of stochasticity of motion in some cases, is the phenomenon of splitting of separatrices of the trajectories. Such an approach used in [37], also confirms the above conclusion that the system (3.2) is stochastic, i.e. it is nonintegrable (see also [38]).

## 5. Higgs Mechanism and Stochasticity. A Phase Transition Disorder - Order in the Classical System [18].

In the recent years, great interest attaches to realization of one or another phase in gauge theories [39-41]: confinement phase - disorder, Higgs phase - order. By analogy, one can say that to the absence of total set of nontrivial (the so-called isolating) integrals in the classical system, there corresponds the disorder phase which the system (3.2) and its generalizations for three degrees of freedom are found to have, while to the systems with a complete set of isolating integrals (when their number is equal to the number of degrees of freedom) there corresponds the order phase.

In connection with aforesaid, the investigation of the classical gauge systems with spontaneous breaking of symmetry seems highly interesting.

Consider the gauge theory with isodoublet breaking of the SU(2) group in the gauge $A_0^a = 0$.

The Hamiltonian corresponding to (2.4) has the form

$$H = H_{YM} + \frac{1}{2}\left(\dot{B}_a^2 + \dot{\sigma}^2\right) + \tag{5.1}$$
$$+ \frac{g^2}{4}\left(A_i^a \; A_i^a\right)\left[\frac{B_a^2}{2} + \left(\frac{\sigma}{\sqrt{2}} + \eta\right)^2\right] +$$
$$+ \lambda^2\left[\frac{B_a^2}{2} + \left(\frac{\sigma}{\sqrt{2}} + \eta\right)^2 - \eta^2\right]^2 ,$$

and the constraint equations are as follows

$$\varepsilon^{abc} A_i^b \dot{A}_i^c - \frac{\eta}{\sqrt{2}}\dot{B}_a + \frac{1}{2}\left[\sigma\dot{B}_a - B_a\dot{\sigma} - \varepsilon^{abc} B_b \dot{B}_c\right] = 0 \tag{5.2}$$

where $\lambda$ is the vacuum expectation value of the scalar field $\varphi$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i B_1 + B_2 \\ \sqrt{2}\,\eta + \sigma - i B_3 \end{pmatrix};$$

$\lambda$ is the self-action coupling constant of the scalar field $\varphi$ .

Let us study in detail the two-dimensional case of the gauge field ((3.1)) that interacts with the Higgs vacuum ( $B_a = \sigma = 0$ ):

$$H \equiv \mu^4 = \frac{1}{g^2}\left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{x^2 y^2}{2} + \frac{g^2\eta^2}{4}\left(x^2 + y^2\right)\right] \tag{5.3}$$

The additional potential energy $\sim x^2 + y^2$ is, naturally, spherically symmetric here too, and again $N^a = 0$.

It is clear that at large fields, this addition, corresponding to the linear oscillator, is inessential, so we must get a random motion of the system (3.2). At small fields, on the contrary, the nonlinearity of $x^2 y^2$ may be thought negligible and stable regular oscillations may be expected.

One can readily be convinced - using the scale transformation $x \to \alpha x$ $y \to \alpha y$ , $t \to \beta t$ - that the motion of the system (5.3) is characterized by the only essential dimensionless parameter $\mathcal{T} = \frac{g^2 \eta^4}{\mu^4}$ .

At $\mathcal{T} = 0$ we, of course, have the stochastic motion investigated in detail in Secs. 3 and 4. At large $\mathcal{T}$ , as was already mentioned, a regular motion is to be expected.

Now we shall see that actually, the system (5.3) is chaotic not only at $\mathcal{T} = 0$, but also at small but finite $\mathcal{T} \leq \mathcal{T}_c$ .

Our purpose is just to calculate on a computer the critical value of parameter $\mathcal{T}_c$ , at which the "phase transition" occurs in the following sense: at large values of $\mathcal{T}$ the system comes close to integrable one and the trajectory in the phase space ( $x, \dot{x}, y, \dot{y}$ ) represents the tore winding [42] (the measure of ergodic trajectories is equal to zero [42], [1], [2] ) , i.e. the phase of order is realized, while for small but finite values of $\mathcal{T}$ ( $\mathcal{T} < \mathcal{T}_c$ ) the motion, just as at $\mathcal{T} = 0$, is stochastic, i.e. the phase of disorder is realized.

In Sec.4 we have already described a part of the computer experiment associated with solving Eqs.(3.2), that lights out the points of intersection of the phase trajectory of the system in the space ( $x, \dot{x}, y, \dot{y}$ ) with the phase plane ( $y, \dot{y}$ ) at $\dot{x} > 0$ .

In Fig.4 we show the photographed from the display of the computer picture in the plane ( $y, \dot{y}$ ) for $\mathcal{T} = 4.84$; one can see that the points of intersection of the trajectory with the plane form closed regular curves. The stable trajectories correspond to the centres of three small closed curves, while the unstable periodic trajectories correspond to two points of intersection of closed lines (the intersection of separatrices at nonzero angle, of which we have already spoken in Sec.4).

Precisely in the vicinity of the two last points of intersection the "macroscopic" regions of the ergodic motion of nonzero measure originally arise (Fig.5, $\mathcal{T} = 0.35$, cf. [33-35], see also [36]).

With a further decrease of $\mathcal{T}$, the area occupied by stochastic component increases sharply, and at a critical value $\mathcal{T} = \mathcal{T}_c \approx 0.15$ becomes almost equal to the whole allowed region of motion on plane ($y, \dot{y}$). The picture then resembles the one shown in Fig.3 (Sec.4) and corresponds to developed stochasticity (we emphasize again that all the points in this Figure correspond to the same trajectory).

A detailed analysis of Higgs mechanism as stabilizing the motion is given in Ref. [36]. For the case with $n = 2$ at small $\mathcal{T}$ ($\mu \gg 1$, $\mathcal{T} << \mathcal{T}_c$) the motion is chaotic ( $h \sim \mu / \ln \mu \sim \frac{\mathcal{T}^{-1/4}}{\ln 1/\mathcal{T}} > 0$ ). At large $\mathcal{T}$ $(\mathcal{T} \gg \mathcal{T}_c)$ the chaotic component is preserved only in the exponentially narrower layer around the separatrix, i.e. the system is integrable in the sense of KAM-theory [43, 42].

Since $h$ depends continuously on $\mathcal{T}$, then the "phase transition" revealed in [18], apparently has a smeared transition region (see also [44], wherein for determination of $\mu_{crit.}$ (or $\mathcal{T}_{crit.}$), the approach based on the study of the topology of the energy surface $H = H(x, y, \dot{x}, \dot{y})$ is applied. At $\mathcal{T} < \mathcal{T}_{crit.} = 2/3$, a gradual transition from regular trajectories to irregular ones is characteristic for our problem. At $\mathcal{T} > \mathcal{T}_{crit.}$, the extraction of a compact invariant manifold filled with regular trajectories is possible).

In passing on to a larger number of degrees of freedom ($n = 3$) [36], the situation becomes more complicated. Even at small $\mu$ ($\mathcal{T} >> 1$), a significant chaotic component is revealed under certain conditions. At large $\mu$ ($\mathcal{T} << 1$), as well as for $n = 2$, the chaotic component covers almost the whole energy surface except for the small regions along the coordinate axes.

Hence we may assert that at $n > 2$, the Higgs mechanism, even at large values of $\mathfrak{J}$, does not eliminate entirely the stochastic component [36].

We now make a remark concerning the treatment of the Yang-Mills-Higgs system (5.3) in the quantum-mechanical limit.

The corresponding problem with the nonlinearity parameter $\alpha$ was treated in [23]:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}(x^2 + y^2) + \alpha x^2 y^2 \tag{5.4}$$

In the classical problem the authors of [23] also observed a transition (at large energies, which, as is readily seen, corresponds to small values of our parameter $\mathfrak{J}$) from regular motion to chaos.

In the quantum problem (5.4), although considering the term with as perturbation, it is found (in agreement with the expectation [21] we spoke of in the Introduction) that there exists a close correlation between the portion of classical random motion and the portion of that part of the energy spectrum of Hamiltonian (5.4), in which the eigenvalues are highly sensitive to small changes in the perturbation parameter (directly corresponding to the absence of intersection of the levels).

The energy region, where a transition from one regime to another is observed, is the same for both classical and quantum cases.

In Ref. [23] it is clearly seen how with increasing energy the fraction of the nondegenerated energy eigenvalues, being strongly dependent on small change of the nonlinearity parameter $\alpha$, increases

$$\Delta_i^2 = \left|\left[E_i(\alpha + \Delta\alpha) - E_i(\alpha)\right] - \left[E_i(\alpha) - E_i(\alpha - \Delta\alpha)\right]\right| \approx$$
$$\approx \left|\frac{\partial^2 E_i}{\partial \alpha^2}\right| (\Delta\alpha)^2$$

and how the portion of levels with $\Delta_i^2 \approx 0$ decreases.

It is interesting to study this problem, not considering the parameter $\alpha$ as perturbation.

At the same time, it must be borne in mind that the quantum mechanical system is not yet a quantum field system with an infinitely large number of degrees of freedom.

And though, as we already mentioned, the homogeneous fields, we are interested in this review, correspond to the long-wave part of the spectrum of the classical Yang-Mills system, the relevant problem in quantum field theory must not ignore the fact that the infrared problem is the problem of strong coupling.

## 6. Classical Yang-Mills Mechanics with $n > 3$.

If, as was shown in the previous sections, the classical Y.M. mechanics (i.e. space-homogeneous fields) is characterized by dynamical stochasticity already at $n = 2, 3$, then the increase of the number of degrees of freedom in the system, generally speaking, must but intensify chaotic nature of motion.

At the same time, the increase of $n$ introduces a new aspect, which we shall consider in this section.

We shall study the system (2.3) for $n = 4$: $A_i^3 = A_3^1 = A_3^2 = 0$

In this case, the third component of conserved moment is nonzero:

$$M_3 = A_1^a \dot{A}_2^a - A_2^a \dot{A}_1^a \tag{6.1}$$

while the constraint condition ( $N^a$ is zero) has the form

$$A_i^1 \dot{A}_i^2 - A_i^2 \dot{A}_i^1 = 0 \tag{6.2}$$

The form of potential $U = \frac{g^2}{4}\left(A_1^1 A_2^2 - A_1^2 A_2^1\right)^2$ hints at a substitution

$$gA_1^1 = \xi_1 + \xi_2 \qquad gA_1^2 = \xi_3 + \xi_4$$
$$gA_2^1 = \xi_2 - \xi_4 \qquad gA_2^2 = \xi_1 - \xi_2, \tag{6.3}$$

that "mixes up" components of different isotopic vectors

$$\vec{A}^{(1)}(A_1^1, A_2^1, 0), \quad \vec{A}^{(2)}(A_1^2, A_2^2, 0)$$

The Hamiltonian takes the form

$$g^2 H = \sum_{i=1}^{4} \dot{\xi}_i^2 - \frac{1}{4}\left(\xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2\right). \tag{6.4}$$

Expressing $M_3$ from (6.1) via variables (6.3) and using the constraint (6.2), we arrive at

$$g^2 \frac{M_3}{4} = \xi_4 \dot{\xi}_1 - \xi_1 \dot{\xi}_4 = \xi_2 \dot{\xi}_3 - \xi_3 \dot{\xi}_2 . \tag{6.5}$$

It is convenient to introduce new variables

$$\xi_1 = \tau_1 \sin\varphi \qquad \xi_2 = \tau_2 \sin\Theta$$
$$\xi_4 = \tau_1 \cos\varphi \qquad \xi_3 = \tau_2 \cos\Theta, \tag{6.6}$$

so that (6.5) will take a simpler form

$$g^2 \frac{M_3}{4} = \tau_1^2 \dot{\varphi} = - \tau_2^2 \dot{\Theta} . \tag{6.7}$$

Substituting (6.6) and (6.7) into (6.4) we shall finally obtain

$$g^2 H = \frac{\dot{\tau}_1^2 + \dot{\tau}_2^2}{2} + \frac{M_3^2}{32}\left[\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right] + \frac{1}{4}\left(\tau_1^2 - \tau_2^2\right)^2 \tag{6.8}$$

Taking $M_3 = 0$ and making substitution $\tau_1 = \frac{x+y}{2}$ , $\tau_2 = \frac{x-y}{2}$ , we shall certainly arrive at the studied in detail Hamiltonian with $n = 2$ (3.1)

Eq.(6.8) was originally obtained in [45] , where, unlike here, not Hamiltonian but axial gauge $A_3^a = 0$ was used.

The author of [46] has recently arrived at (6.8) , however he didn't notice the constraint condition, which leads to the equality $L_1 = L_2$ of two constants that play a role of our $M_3$ ( $= L_1 = L_2$ ).

Although everywhere in [46], just the case $L_1 = L_2$ is considered (as an assumption).

The obtained system with Hamiltonian (6.8) is actually independent of parameter $M_3$ , for by means of the scale transformations

$$\tau_1 \to \left(\frac{M_3}{4}\right)^{1/3} \tau_1 , \quad \tau_2 \to \left(\frac{M_3}{4}\right)^{1/3} \tau_2 , \quad t \to \left(\frac{M_3}{4}\right)^{-1/3} t \quad ,$$

$H$ can be transformed as follows [45] :

$$g^2 H = \left(\frac{M_3}{4}\right)^{4/3} \left[ \frac{\dot{\tau}_1^2 + \dot{\tau}_2^2}{2} + \frac{1}{2}\left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right) + \right. \\ \left. + \frac{1}{4}\left(\tau_1^2 - \tau_2^2\right)^2 \right] . \tag{6.9}$$

The corresponding equations of motion have the form

$$\begin{aligned} \ddot{\tau}_1 &= \frac{1}{\tau_1^3} + \tau_1\left(\tau_1^2 - \tau_2^2\right) \\ \ddot{\tau}_2 &= \frac{1}{\tau_2^3} - \tau_2\left(\tau_1^2 - \tau_2^2\right) \end{aligned} \tag{6.10}$$

The region $\tau_1 = \tau_2 = 0$ is forbidden because of the centrifugal barrier in (6.9).

The trajectories of the system in each quadrant of the coordinate system $\tau_1$ , $\tau_2$ lie inside the region bounded by the equipotential curve shown in Fig.6.

Here also, as it was the case with the system (3.2), the same picture of a "random walk" of the particle in the channel along the quadrant bisectrix, is again reproduced qualitatively. Ruling out the trivial case, when the velocity of the particle is directed precisely along the channel axis, the particle, as the analysis shows, cannot, generally speaking, go to infinity, for the channel's width reduces with time more rapidly (like $t^{-2}$) than does the amplitude of oscillations near the channel's axis ($\sim t^{-1/2}$) [45]. This circumstance is, in essence, responsible for the stochasticity which is particularly pronounced at not small values of $H = \mu^4$.

We will not dwell on the details of the analysis of this stochasticity and the question of regular component, referring the interested readers to the works [45, 47, 48]. We shall make only one remark.

In Sec.5 we observed a stabilizing action of the Higgs mechanism on the system (3.2). It actually took place owing to introduction into the system (3.2) of additional parameter $\pi$, which, when varied, changed at a given $\mu$ the regime of motion. The system (6.8), despite nonzero moment $M_3$, has no such parameter. However, if introducing into the right-hand side of classical Y.M. equations the density of charge $J_\mu^a = (\rho^a, 0)$ (which is, in essence, of quantum origin and generated, for example, by heavy virtual quarks) $\rho^a = \rho\,\varepsilon^{abc} n_1^b n_2^c$, where $n_i^a$ are unit vectors in inner space, then the motion integrals $M_3$ and $N$ will take the form

$$M_3 = r_1^2 \dot{\varphi} - r_2^2 \dot{\theta}$$

$$\frac{\rho}{2} = r_1^2 \dot{\varphi} + r_2^2 \dot{\theta}$$

Then, instead of (6.9), we shall have 48 (after scale transformation $\tau_1 \to \left(\frac{\mu+\rho}{4}\right)^{1/3} \tau_1$ and so on):

$$g^2 H = \frac{\dot{\tau}_1^2 + \dot{\tau}_2^2}{2} + \frac{1}{2}\left(\frac{1}{\tau_2^2} + \frac{\lambda^2}{\tau_1^2}\right) + \frac{1}{4}\left(\tau_1^2 - \tau_2^2\right)^2, \tag{6.11}$$

where in the system there appeared, besides energy, a new parameter

$$\lambda = \frac{\mu - \rho}{\mu + \rho}$$

A numerical investigation carried out in [48] reveals that at $\lambda \neq 1$ at a given $H$, a transition from stochastic motion to a regular one takes place as $\lambda$ increases. At $\lambda > \lambda_{crit.}(H)$ the system is close to the regular one. Of course, the stabilizing action associated with the charge density $\rho^a (\lambda \gg 1)$ is of absolutely different nature that that connected with the Higgs mechanism.

## 7. The General Case of Classical Y.M. Mechanics.

In the work of G.Asatryan and G.Savvidy [49], the most general case of classical Y.M. mechanics with nine degrees of freedom is considered.

The potential of Y.M. field $A_i^a(t)$ can always be presented as

$$\left(O_1 \, E \, O_2^T\right)_i^a \tag{7.1}$$

where $E$ is diagonal

$$E = \begin{pmatrix} x(t) & 0 & 0 \\ 0 & y(t) & 0 \\ 0 & 0 & z(t) \end{pmatrix} \tag{7.2}$$

and $O_1$ and $O_2$ are orthogonal time-dependent matrices.

Introducing antisymmetric matrices $\omega$ and $\Omega$

$$\omega = O_1^T \dot{O}_1 = -\dot{O}_1^T O_1 \tag{7.3}$$

$$\Omega = O_2^T \dot{O}_2 = -\dot{O}_2^T O_2$$

we obtain for Hamiltonian

$$H_{YM} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + T_{YM} + \tag{7.4a}$$

$$+ \frac{g^2}{4}\left(x^2 y^2 + y^2 z^2 + z^2 x^2\right)$$

where

$$T_{YM} = \frac{1}{2}\sum_{a=1}^{3}\left\{ I_a\left(\omega_a^2 + \Omega_a^2\right) - 2 J_a \omega_a \Omega_a \right\} \tag{7.4b}$$

$$\omega_a = \frac{1}{2}\varepsilon_{abc}\,\omega_{bc}\,, \quad \Omega_i = -\frac{1}{2}\varepsilon_{ijk}\,\Omega_{jk} \tag{7.5}$$

$$I_1 = y^2 + z^2, \quad I_2 = x^2 + z^2, \quad I_3 = x^2 + y^2 \tag{7.6}$$

$$J_1 = 2yz\,, \quad J_2 = 2xz\,, \quad J_3 = 2xy\,.$$

If the analogy with classical mechanics of the point was helpful in the previous sections, an analogy with mechanics of a solid body is useful here. Only, this "body" has $t$ dependent inertia moments $J_i$ and $I_i$ and "rotates" in usual and inner spaces, for if we project the moments $M_i$ and $N^a$ onto the coordinate system "moving" together with the "body"

$$N^a = O_1^{ab}\, n^b\,, \qquad M_i = O_{2\,ij}\, m_j$$

$$N^a = I_a \omega_a - J_a \Omega_a\,, \quad M_k = I_k \Omega_k - J_k \omega_k\,,$$

and differentiate them with respect to time ( $\dot{N}^a = \dot{M}_i = 0$), we shall then arrive at the "Euler equation" of classical mechanics

$$\frac{d\vec{n}}{dt} = [\vec{n}, \vec{\omega}], \quad \frac{d\vec{m}}{dt} = [\vec{m}, \vec{\Omega}]. \tag{7.7}$$

We shall not analyze in detail these equations (7.7) and (7.4), but refer the reader to the original work [49] , where also the generalization for the case of an arbitrary gauge group SU(N) is available.

## 8. Stochasticity and Confinement.

The discovery of dynamical chaos of free classical non-Abelian gauge fields and of the "phase transition" of the "disorder - order" type ("confinement phase" - "Higgs phase") in these systems, which were the contents of the previous sections, makes highly attractive the idea that the observed phenomena are to a certain extent preserved in the real (i.e. quantum) vacuum of QCD and that precisely the presence of random color vacuum fields in it is responsible for the color confinement.

We have already mentioned in the Introduction the arguments in favour of that the disordered (stochastic) vacuum may be the reason of confinement (the analogy with lowering of dimensionality of quantum spin systems in random field, the Olesen's hypothesis on the reduction of the 4-dimensional Y.M. theory to the two-dimensional one at $N \to \infty$ , the lattice calculations that make this hypothesis plausible also for the SU(2)-symmetry).

We now consider a little in detail the last argument, associated with the Monte-Carlo calculations on the lattice distribution $\rho_c(\alpha)$ of eigenvalues of the Wilson loops $\langle W(C) \rangle = \int_{-\pi}^{\pi} d\alpha \langle \rho_c(\alpha) \rangle e^{i\alpha}$ , for they show that stochastic phenomena are to some degree present at

QCD [15] .

It follows from the results of Ref. 15 that for the distances ( the sizes of loops) $\tau$ less than the confinement radius $\tau_c$ , the distribution of eigenvalues of loops (the spectral density) $\rho_c(\alpha)$ has a peak at $\alpha = 0$, that means strong correlation of fields at small distances.

However for loops with $\tau \gtrsim \tau_c$ , the distribution $\rho_c(\alpha)$ becomes practically homogeneous, i.e. the fields are weakly correlated and the distribution of eigenvelues of $W(C)$ corresponds to the disordered configurations.

This argument, if not being an artifact of the Monte-Carlo lattice calculations, idicates that stochastic component in some form is indeed present at QCD.

The question of whether this stochasticity is the manifestation and "relic" of the observed and described in Secs. 1-7 classical stochasticity, is left, of course, open.

At present, there is a large number of mechanisms that "provide" the color confinement, the most popular among which is the mechanism based on condensation of vortices and magnetic monopoles [40 , 50, 51] . No less popular is the mechanism [52] based on the idea of lowering the vacuum energy owing to occurrence of gluonic condensate [28, 29] . The Monte-Carlo calculations [53-55] show that quarks are confined in the SU(2) and SU(3) gauge lattice theories.

As concerning less orthodox mechanisms of confinement, we mention the work of Kirzhnits et al. [56] , where the above-discussed possible stochasticity of QCD is associated with the phenomenon of the type of localization in disordered systems.

The localization due to random potential leads, as is well known, to unusual properties of spectrum of the relevant quantum problem: the spectrum is quasi-continuous (of the type of a set of rational numbers), however the wave functions corresponding to closest energetic levels are localized at a large distance from each other. Therefore, to such localized wave functions corresponds a discrete spectrum, whose levels are determined by the properties of random potential that acts between quarks (in particular, by the localization length).

Thus, the qualitative arguments lead to that in the "quark-antiquark" system in one-dimensional approximation there acts a linear effective potential increasing with distance.

The above-said, with account of the mentioned in the Introduction works [13, 14], wherein the confinement occurred in the limit $N \to \infty$ as a result of stochasticity, is extremely important to exhibit stochasticity as a really sufficient condition for confinement in quantum field theory.

We shall demonstrate [57] that if in the functional integral of theory one takes into account only fields generated by randomly distributed currents, then the appropriate two-particle Green's function will correspond to confinement. The contribution into the gluon propagator is, of course, not exhausted with these fields, for there undoubtedly exists an important class of fields of different, nonstochastic nature, which are particularly important for small and intermediated (as compared to the confinement radius $r_c$ or, in our problem, to the radius of correlation of random currents $\mu^{-1}$) distances. Precisely these fields must be responsible for the asymptotic freedom.

We cannot say, whether they are present at $r \sim \mu^{-1} \sim r_c$ together with the stochastic component (at which the above-mentioned Monte-

Carlo calculations indicate [15] ), and if they are, then what their relative contribution compared to random fields is.

In other words, we are interested in the question: what the Green's function of field quanta is, if the latters are generated by random color currents $J_\mu^a(x)$ which are Gaussian-distributed (the so-called "white noise"):

$$\langle J_\mu^a(x) J_\nu^b(y) \rangle = \mu^2 \delta_{\mu\nu} \delta^{ab} \delta^{(\ )}(x-y). \tag{8.1}$$

In agreement with the abovesaid, the dynamics of quanta of field ("gluons") at the distances of the order of $\tau_c \sim \mu^{-1}$ is determined by the stochastic equation of motion

$$\frac{\delta S}{\delta A_\mu^a} = J_\mu^a \tag{8.2}$$

where $S$ is the action of theory in the four-dimensional space - time (below we shall use the Euclidean formulation), and quantum averaging is determined by the relation

$$\langle A_{\mu_1}^{a_1}(x_1) \dots A_{\mu_n}^{a_n}(x_n) \rangle = \langle \tilde{A}_{\mu_1}^{a_1}(x_1) \dots \tilde{A}_{\mu_n}^{a_n}(x_n) \rangle_J . \tag{8.3}$$

The fields $\tilde{A}_\mu^a$ are determined from Eq.(8.2), and the averaging $\langle \dots \rangle_J$ in the right-hand side of Eq.(8.3) is made over the Gaussian distribution of currents

$$\exp. \left\{ -\frac{1}{2\mu^2} \int J_\mu^a(x) J_\mu^a(x) d^4x \right\}$$ , corresponding to (8.1).

The generating functional of our theory, corresponding to (8.3), is given by the expression

$$Z(h_\mu^a) = \int \mathcal{D}J \exp\left\{-\int\left[\frac{1}{2\mu^2} J_\mu^a(x^2) - \right.\right. \tag{8.4}$$

$$\left.\left. - h_\mu^a(x)\,\tilde{A}_\mu^a(x)\right] d^4x\right\}$$

which, being differentiated with respect to the quantum source $h_\mu^a(x)$, defines the Green's functions (8.3).

Passing in (8.4) from the variables $J$ to $\tilde{A}$ by introducing the $\delta$-function, that corresponds to (8.2), and applying the standard procedure of rewriting the arising determinant via the anticommuting vector fields $\psi_\mu^a$, $\bar{\psi}_\mu^a$, we shall arrive at the formula (for simplicity, we shall omit the Lorentz and internal indices below):

$$Z(h) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A \exp\left\{-\int\left[\frac{1}{2\mu^a}\left(\frac{\delta S}{\delta A}\right)^2 \delta(x-y) - \right.\right.$$

$$\left.\left. -\bar{\psi}(x)\,\frac{\delta^2 S}{\delta A(x)\,\delta A(y)}\,\psi(y) - h\,A(x)\,\delta(x-y)\right] d^4x\, d^4y. \tag{8.5}$$

Note that the introduction of the stochastic equation (8.2) into the functional integral is notable by that we do not introduce the auxiliary time as the fifth component, but operate in the real space - time. We here observe a close relation between the stochastic differential equations of the type of (8.2) with supersymmetry, which was noticed recently [58].

One can see already from (8.5) that in the tree approximation the two-particle function of the type of (8.3) has the properties of confinement (see the first term in the exponent (8.5)).

We shall show this, however, using another method, associated with the introduction of superfield $\Phi_\mu^a(x,\theta)$:

$$\Phi_\mu^a(x,\theta) = A_\mu^a(x) + \bar{\psi}_\mu^a(x)\,\theta + \bar{\theta}\,\psi_\mu^a(x) + C_\mu^a\,\bar{\theta}\,\theta, \tag{8.6}$$

here $\theta, \bar{\theta}$ are the anticommuting variables ( $\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0$ ).

One may easily be convinced that (8.5) can be written in the form

$$Z(h) = \int \mathcal{D}\Phi(x,\theta)\, \exp\Big\{ -\int \Big[ \mathcal{L}(\Phi) - \frac{\mu^2}{2}\Phi^+ \frac{\partial^2}{\partial\bar{\theta}\partial\theta}\Phi - H\Phi \Big] d^4x\, d\bar{\theta}\, d\theta \tag{8.7}$$

where $H = h(x)\bar{\theta}\theta$.

From (8.7) it follows that the Fourier transform of the propagator $\langle \Phi^a_\mu \Phi^b_\nu \rangle$ of superfield has the structure of ( $p^2 \lesssim \mu^2$ ):

$$(p^2 + \mu^2 \bar{\alpha}\alpha)^{-1} \delta^{ab} \delta_{\mu\nu}$$

where $\bar{\alpha}, \alpha$ are Grassman's variables corresponding (after the Fourier transform) to the variables $\bar{\theta}, \theta$ .

Integrating over them we shall arrive at confinement, for the Fourier transform $\langle A^a_\mu A^b_\nu \rangle$ has finally the form

$$\delta_{\mu\nu}\, \delta^{ab} \mu^2 / p^4$$

i.e. the exchange of such quanta is responsible for the linearly increasing with $\tau$ (for $\tau \gtrsim \tau_c$ ) potential between the static sources.

The consideration we have just performed, shows that the stochasticity of the sources generating the corresponding fields (and they may occur, as the Monte-Carlo calculations show [15] , in the vicinity of $\tau \gtrsim \tau_c$ ) is the sufficient condition for the linear potential.

One may be convinced that stochasticity is the necessary condition as well, if based on local field theories. However, the reader has apparently noticed that such confinement is not something extraordinary for gauge theories, for, as is seen from our conclusion, any quantum field theory, that is characterized by the condition (8.1) of stochasticity (the restriction by the "white noise" as an example of stochasticity is apparently insignificant), will have propagator with behaviour $\mu^2/p^4$ at $p^2 \lesssim \mu^2$ .

This is well seen from the following chain of the symbolic equations and relations:

$$\Box A = - J, \qquad A = - \Box^{-1} J,$$

$$\langle A(x) A(y) \rangle = \Box_x^{-1} \Box_y^{-1} \langle J(x) J(y) \rangle =$$

$$= \mu^2 \Box_x^{-1} \Box_y^{-1} \delta^{(4)} (x - y)$$

and so on, from where our statement follows, owing precisely to the correlator (8.1).

What then makes the gauge theory of non-Abelian fields unique as compared with the other theories?

Apparently, the inherent in this theory dynamical stochasticity, which we have considered in detail in this paper for the classical case. We have also advanced arguments in favour of that stochastic phenomena are probably preserved in the quantum case, too. Of course, further explorations are needed here.

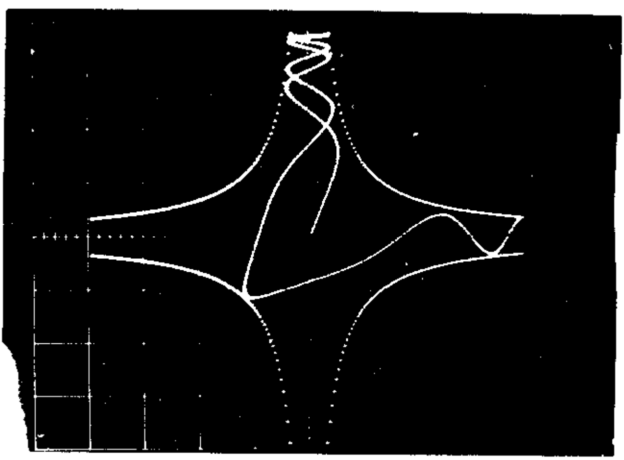

Fig.1

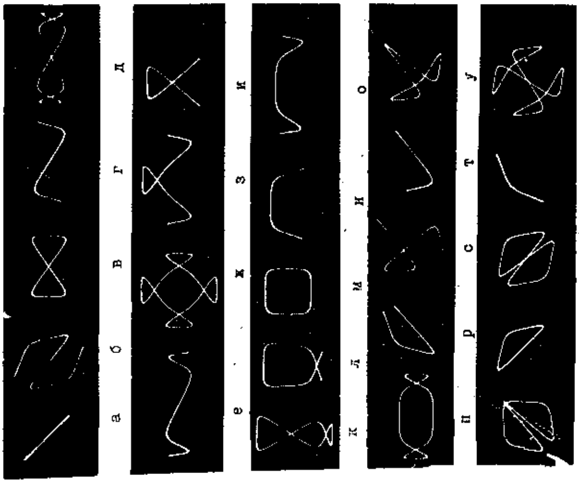


Fig.2

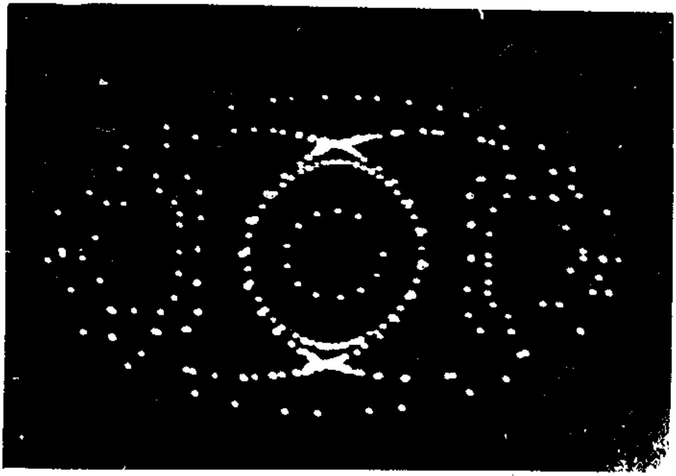

Fig.4

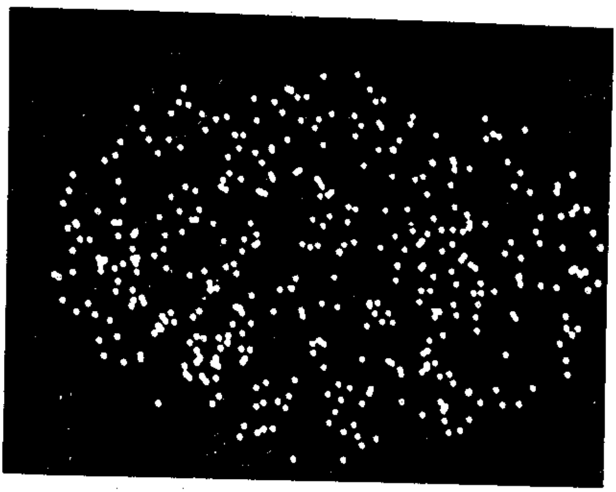

Fig.3

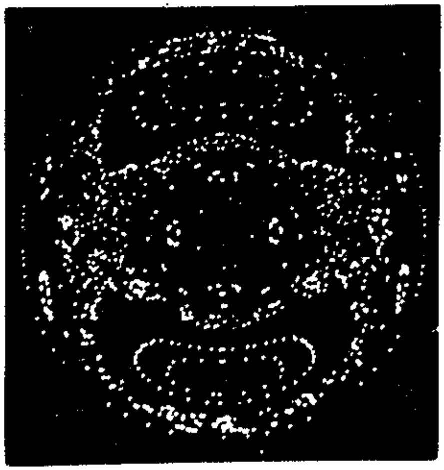

Fig.5

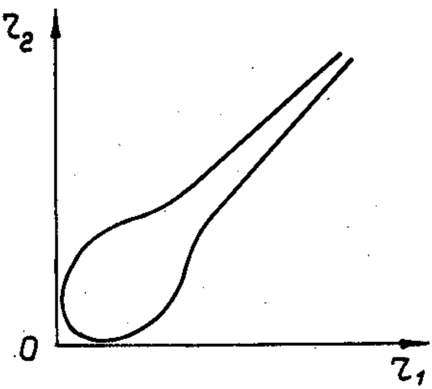


Fig.6

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