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COLLECTIVE RELAXATION OF STELLAR SYSTEMS

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COLLECTIVE RELAXATION OF STELLAR SYSTEMS

The problem of relaxation of the stellar systems is investigated from the concept of the ergodic theory which essentially takes into account the collective nature of gravitational interaction. It is shown that an exponential instability exists in the stellar systems leading to the equilibrium state. The relaxation time for the real stellar systems is calculated. That time is essentially less than binary (Chandrasekhar) relaxation time. Their physical differences is discussed. The obtained results throw light onto the nature of the rich variety of stellar configurations observed in the Universe.

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КОЛЛЕКТИВНАЯ РЕЛАКСАЦИЯ ЗВЕЗДНЫХ СИСТЕМ

Исследуется проблема релаксации звездных систем с позиций эргодической теории, существенно учитывающей коллективный характер гравитационного взаимодействия. Показано, что звездные системы обладают экспоненциальной неустойчивостью, приводящей к равновесному состоянию. Вычислено время релаксации для реальных звездных систем. Это время существенно меньше времени релаксации с учетом только двойных столкновений (чандрасекаровское). Обсуждается их физическое отличие. Полученные результаты позволяют понять природу разнообразия звездных конфигураций наблюдаемых во Вселенной.

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1. Introduction

The real stellar systems - globular clusters and galaxies - are known to be generally in the equilibrium state. This is reflected in the high degree of regularity of basic physical characteristics of the stellar systems, i.e. surface luminosity, dispersion of velocities, geometric shapes, etc.

Jeans, Schwarzschild, Eddington, Ambartzumian, Chandrasekhar, Spitzer and others have formulated many fundamental principles of statistical mechanics of the stellar systems. Thus, Chandrasekhar [1] has considered in detail the relaxation mechanism based on the account of the most natural process - stellar binary encounters. However the magnitude of the relaxation time of the real stellar systems (especially elliptical galaxies), owing only to binary encounters, turned out to be more than 10^{13} years, i.e. exceeded the Hubble time. This contradiction is known as Zwicky paradox.

An important step to eliminate this paradox became Lynden-Bell's paper [2] in which the theory of the collisionless violent relaxation was developed. This theory, having great heuristic advantages and stimulating a lot of papers, simultaneously could not avoid certain difficulties. Being a theory describing essentially non-equilibrium phase of evolution of the stellar systems, it cannot describe their quasi-equilibrium phase. Among many further attempts as to understand the mechanics of collisionless stellar systems, it is worth mentioning the paper by Severne and Luwel [3], wherein the contribution of fluctuations of the self-consistent field to the relaxation process has been considered.

The interest to the relaxation problem and the dynamical evolution of the stellar systems has grown sharply due to a number of recently obtained interesting observational data. Thus, at the centre of the globular cluster M15, an anomalous excess of brightness is observed, whose explanation by the presence of the central black hole [4] encounters certain difficulties ([5], see also [6]). Existence of a "rapid" mechanism of relaxation could become the possible explanation of this fact.

The problem of the shape of elliptical galaxies has acquired a new content (see the recent review [7]).

The difficulties in satisfactory understanding of dynamics of the stellar systems are due to the well-known fact that in a system of N gravitationally interacting stars Debye screening, as distinct from plasma, is absent. This circumstance makes the statistical description of gravitational systems more complicated and requires special methods.

All that point out the crucial role of collective effects in the process of relaxation of stellar systems.

The present study is aimed at the investigation of this problem from the viewpoint of the ergodic theory which essentially takes into account the collective nature of interaction.

In the ergodic theory [8-11] a great progress is achieved in the investigation of statistical properties of dynamical systems prescribed by differential equations. The methods developed in the ergodic theory have found an application in the investigations of N.S.Krylov [12] , who had originally used these methods for the description of the relaxation process.

One of the authors (GKS) [13, 14] recently used these methods studying the statistical properties of non-Abelian gauge Yang-Mills fields being the basis of the theory of elementary particle interaction. That system was proved to be Kolmogorov K-system, i.e. non-integrable, possessing exponential instability and strong statistical properties.

The classification of non-integrable dynamical systems by the increasing of degree of their statistical properties is obtained in the ergodic theory. K-systems [15] possess maximally strong statistical properties. These systems tend to the equilibrium state with exponential rate, and the index of exponent is naturally considered as the relaxation time.

As is shown in this paper, the stellar systems possess exponential instability peculiar to K-systems; the relaxation time is calculated, which is connected with the Chandrasekhar (binary) relaxation time by the relation (see (5.21))

$$\tau \sim \frac{R_*}{d} \tau_{Ch} \tag{1.1}$$

where R_* is the radius of gravitational influence of the star, d is the mean distance between the stars. So far as for the real stellar systems $d \gg R_*$, then

$$\tau \ll \tau_{Ch} \tag{1.2}$$

The time τ is substantially less than the Hubble time.

2. Reduction of the N-Body Problem to the Study of Geodesic Flux in the 3N-Dimensional Riemann Manifold

Denote by $\vec{r}_\alpha \ (\alpha = 1, \dots, \mathcal{N})$ the coordinates of stars. The potential of interaction is

$$U = \sum_{\alpha<\beta} U(\vec{r}_\alpha - \vec{r}_\beta) = -G \sum_{\alpha<\beta} \frac{M_\alpha M_\beta}{r_{\alpha\beta}} ; \qquad \vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta , \tag{2.1}$$

where M_α is the stellar mass.

The equations of motion in the Hamiltonian form are

$$\dot{\vec{P}}_\alpha = -\frac{\partial H}{\partial r_\alpha} ; \qquad \dot{\vec{r}}_\alpha = \frac{\partial H}{\partial \vec{P}_\alpha} , \tag{2.2}$$

where H is the complete Hamiltonian of the system, $\vec{P}_\alpha$ is the star momentum. So far as H is explicitly time-independent, then $H(\vec{p}, \vec{r})$ is an integral of motion, while the equation $H(\vec{p}, \vec{r}) = E = const$ determines the $6\mathcal{N} - 1$ dimensional energy hypersurface in 6N-dimensional phase space.

By means of the variational principle of Maupertuis the trajectories of motion of the system (2.2) may be presented as geodesics of some Riemann metric given in the region of the configurational space $(\vec{r}_1, \dots, \vec{r}_\mathcal{N}) \in Q$ if $U(\vec{r}_\alpha) < E$

$$ds^2 = (E - U) d\rho^2 = W \sum_{\alpha=1}^{3\mathcal{N}} (dq^\alpha)^2 , \tag{2.3}$$

where $\{q^\alpha\}$ is defined as

$$\left\{ q^{\alpha} \right\} = \left\{ M_1^{1/2} \vec{r}_1 , \ \ldots \ , M_{\mathcal{N}}^{1/2} \vec{r}_{\mathcal{N}} \right\} , \tag{2.4}$$

$$\alpha = 1, \ \ldots \ , 3\mathcal{N} .$$

The main idea is that the study of the behaviour of the geodesics, and hence the trajectories of the system (2.2) reduces to the investigation of geometrical properties of the Riemann manifold prescribed by the metric (2.3).

The equation of the geodesics on the Riemann manifold

$$\frac{d^2 q^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dq^{\beta}}{ds} \frac{dq^{\gamma}}{ds} = 0, \tag{2.5}$$

($\Gamma^{\alpha}_{\beta\gamma}$ are the Cristoffel symbols) for metric (2.3), where

$$g_{\alpha\beta} = W\delta_{\alpha\beta} , \ g^{\alpha\beta} = \frac{1}{W} \delta_{\alpha\beta} , \tag{2.6}$$

has the form

$$\frac{d^2 q^{\alpha}}{ds^2} + \frac{1}{2W}\left[2 \frac{\partial W}{\partial q^{\gamma}} \frac{dq^{\gamma}}{ds} \frac{dq^{\alpha}}{ds} - g^{\alpha\gamma} \frac{\partial W}{\partial q^{\gamma}} g_{\beta\gamma} \frac{dq^{\beta}}{ds} \frac{dq^{\gamma}}{ds} \right] = 0 . \tag{2.7}$$

Eq.(2.7) coincides with equations of motion (2.2) if the eigentime ds is replaced by $\sqrt{2} \ W dt$.

The global properties of the geodesics are defined by the linear deviation δq of close geodesics which satisfies the equation

$$\frac{D^2 \delta q^{\alpha}}{Ds^2} = - R^{\alpha}_{\beta\gamma\delta}(q) \frac{dq^{\beta}}{ds} \delta q^{\gamma} \frac{dq^{\delta}}{ds} , \tag{2.8}$$

where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann tensor, D/Ds denotes the covariant derivative.

From (2.8) for the magnitude

$$|\delta q|^2 = g_{\alpha\beta}\,\delta q^{\alpha}\,\delta q^{\beta} \tag{2.9}$$

one can readily obtain the following equation

$$\frac{d^2|\delta q|^2}{ds^2} = -2R_{\alpha\beta\gamma\delta}(q)\,\delta q^{\alpha}\,\frac{dq^{\beta}}{ds}\,\delta q^{\gamma}\,\frac{dq^{\delta}}{ds} + 2\left|\frac{D\,\delta q}{Ds}\right|^2 \tag{2.10}$$

from which one can see that the linear deviation of the geodesics depends on the geometry of the Riemann manifold determined by the Riemann tensor.

Indeed, by definition, the curvature along two-dimensional directions δq and dq/ds is K :

$$K(\delta q, dq/ds)\cdot|\delta q \wedge dq/ds| = R_{\alpha\beta\gamma\delta}\,\delta q^{\alpha}\,\frac{dq^{\beta}}{ds}\;\;\delta q^{\gamma}\,\frac{dq^{\delta}}{ds}\;;$$

(2.11)

$$|\delta q \wedge dq/ds| = (g_{\alpha\beta}\delta q^{\alpha}\delta q^{\beta})\left(g_{\alpha\beta}\,\frac{dq^{\delta}}{ds}\;\;\frac{dq^{\gamma}}{ds}\right) -$$

$$-\left(g_{\alpha\beta}\,\delta q^{\alpha}\,\frac{dq^{\beta}}{ds}\right)^2\;,$$

and if it is negative in all directions δq and dq/ds , then the linear deviation will change by exponential rate (for details, see Secs. 5,6, Eqs. (5.8, 5.12, 6.2, 6.3))

$$|\delta q(s)| \geqslant |\delta q(0)|\,e^{\sqrt{2K}s} \qquad \text{if } \left.\frac{d|\delta q|}{ds}\right|_{s=0} > 0$$

$$|\delta q(s)| \leqslant |\delta q(0)|\,e^{-\sqrt{2K}s} \qquad \text{if } \left.\frac{d|\delta q|}{ds}\right|_{s=0} < 0 \tag{2.12a}$$

where

$$K = \min_{(\delta q,\, dq/ds)} \left| K(\delta q,\, dq/ds) \right| . \tag{2.12b}$$

The next section deals with the statistical properties of dynamical systems having exponential instability (2.12a) from the viewpoint of the ergodic theory, and their relation with the relaxation time.

3. Statistical Properties of the Dynamical Systems: Definition of the Relaxation Time

The dynamical systems can be divided into two classes - integrable ones, i.e. when the number of conserved integrals is equal to the number of degrees of freedom, and phase trajectories perform winding of N-dimensional torus, and non-integrable ones*, when the phase trajectory covers chaotically the 2N-1 -dimensional energy hypersurface. The classification of non-integrable dynamical systems is given in the ergodic theory [8,9] by the increasing degree of their statistical properties. Those are: the ergodic systems, the systems with weak mixing, with mixing, with n-fold mixing, and finally with K-mixing, i.e. K-systems. Already the systems with mixing have the property of tending to the equilibrium state [10] .

K-systems possess maximally strong statistical properties. One of their main properties is the decay of trajectories in the phase space into beams of exponentially approaching and expanding trajectories (transversal fibers) [16] . Therefore K-systems tend to equilibrium state with exponential rate.

* The non-integrability of three body problem was proved by Poincare in 1892.

It is natural to consider the magnitude equal to the exponent index as the relaxation time.

It is important to establish what class of non-integrable systems the stellar systems belong to.

In [17-19, 12, 16, 20] and other studies rather general criteria were obtained answering the question to what class the systems with given Hamiltonian belong. It was proved, in particular, that the geodesic flux on the Riemann manifold of the variable negative curvature is K-system, so that the curvature determines the exponent index. The physical aspects of the ergodic theory mentioned above are treated in more details in [13] .

Thus, the negativity of the curvature of the Riemann manifold prescribed by metric (2.3) is a sufficient condition for the exponential instability in the stellar systems.

4. Sign of the Scalar Curvature

The Riemann tensor $R_{\alpha\beta\gamma\delta}$ for metric (2.2) has the form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2W}\left[W_{\beta\gamma} g_{\alpha\delta} - W_{\alpha\gamma} g_{\beta\delta} - W_{\beta\delta} g_{\alpha\gamma} + W_{\alpha\delta} g_{\beta\gamma}\right] \tag{4.1}$$

$$- \frac{3}{4W^2}\left[W_\beta W_\gamma g_{\alpha\delta} - W_\alpha W_\gamma g_{\beta\delta} - W_\beta W_\delta g_{\alpha\gamma} + W_\alpha W_\delta g_{\beta\gamma}\right]$$

$$+ \frac{1}{4W^2}\left[g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta}\right] W_\sigma W^\sigma ,$$

and the scalar curvature is

$$R = 3N(3N-1)\left[-\frac{\Delta W}{3N W^2} - \left(\frac{1}{4} - \frac{1}{2N}\right)\frac{(\nabla W)^2}{W^3}\right] , \tag{4.2}$$

where

$$W_\alpha = \frac{\partial W}{\partial q^\alpha}\,; \quad \Delta W = \frac{\partial^2 W}{\partial q^\alpha \partial q^\alpha}\,; \quad (\nabla W)^2 = \frac{\partial W}{\partial q^\alpha} \; \frac{\partial W}{\partial q^\alpha} \;.$$

Let us calculate quantities in (4.2) for the scalar curvature. Taking into account (2.4) and the relation

$$\Delta_{\vec r}\left(\frac{1}{|\vec r - \vec a|}\right) = -4\pi\delta(\vec r - \vec a) \tag{4.3}$$

for the term with ΔW in (4.2) we obtain

$$\frac{3\mathcal{N}(3\mathcal{N}-1)}{3\mathcal{N} W^2}\, 4\pi G \sum_{a\neq b}^{\mathcal{N}} M_a M_b\, \delta(\vec r_a - \vec r_b)\;. \tag{4.4}$$

The expression (4.4) is evidently zero in all cases except those when any two or more stars undergo direct impact. Since the real stellar systems are collisionless (i.e. the time interval between two impacts is very large), one can neglect direct impacts. Therefore, for R we arrive at

$$R = -\frac{3\mathcal{N}(3\mathcal{N}-1)}{W^3}\left(\frac{1}{4} - \frac{1}{2\mathcal{N}}\right)(\nabla W)^2\;. \tag{4.5}$$

Note that neglecting the first term in (4.2) and speaking on the absence of direct impacts, we nevertheless do not ignore the collisions in the Chandrasekhar sense, i.e. close encounters of the stars, as a result of which the direction of their motion changes by a certain angle. Thus in (4.5) the effect of binary stellar encounters as well as possible collective effects is taken into account.

One can see from (4.5) that the sign of R depends on the sign of W and the value of $\mathcal{N}$. At $\mathcal{N}=2$, $R=0$, and this reflects the fact that the problem of two bodies is integrable. At $\mathcal{N} \geqslant 3$ and

$W>0$ we arrive at the important conclusion that R is negative and the system may have exponential instability and is a candidate into K-systems. More precise statements will be done in analysing the sign of two-dimensional curvature (2.11) in Sec.6. But already now we can claim that the stellar systems may be attributed to the class of systems with strongly developed statistical properties.

The conclusion on non-integrability of the N-body problem at $\mathcal{N} \geqslant 3$ agrees with the mentioned result of Poincare.

5. Estimation of the Collective Relaxation Time

According to the results of the previous section, the stellar systems may possess exponential instability, and the exponent index determined by a minimum modulus of the curvature tensor along all two-dimensional directions (2.12) can naturally be considered as the relaxation time.

It is helpful, first, to estimate the mean scalar curvature (4.5). One can readily see that

$$\left(\nabla W\right)^2 = \sum_{a=1}^{\mathcal{N}} \frac{1}{M_a}\left(\frac{\partial U}{\partial \vec{r}_a}\right)^2 = \sum_{a=1}^{\mathcal{N}} M_a \vec{\mathcal{E}}_a^{\,2} , \tag{5.1}$$

where $\vec{\mathcal{E}}_\alpha$ is the field strength.

To estimate the quantities in $R-W$ and ΔW we shall proceed from the virial theorem and the Holtsmark probability distribution for the force [21-23] *.

According to the virial theorem,

$$\bar{W} = E - \bar{U} = -E , \tag{5.2}$$

i.e., as was already mentioned, $\bar{W} > 0$ for the bound systems,

Making use of the Holtsmark distribution, we shall find mean square force affecting a single star

$$\begin{aligned} \langle \vec{\mathcal{E}}^2 \rangle &= \int \vec{\mathcal{E}}^2 \mathcal{H}(\varepsilon)\, d\vec{\mathcal{E}} = \\ &= a^{4/3} \int_0^y y^2 dy \frac{2}{\pi y} \int_0^\infty e^{-\left(\frac{x}{y}\right)^{3/2}} x \sin x \, dx = \\ &= a^{4/3} \int_0^\infty H(y)\, y^2 dy = c a^{4/3} , \end{aligned} \tag{5.3}$$

where $\mathcal{H}$ is the Holtsmark distribution, and

$$a = \frac{4}{15}\left(2\pi G\right)^{3/2} \langle M^{3/2} \rangle n , \tag{5.4}$$

---

* The fact that the stellar systems possess exponential instability may justify the Holtsmark-Chandrasekhar-von Neumann probability approach.

n is the stellar density, $\langle M \rangle$ is the star mean mass.

From (5.1 - 5.4) we have

$$(\nabla W)^2 = \mathcal{N} c a^{4/3} \langle M \rangle . \tag{5.5}$$

Substituting (5.2), (5.5) into (4.5) we shall get the desirable expression for

$$R = \frac{3\mathcal{N}^2(3\mathcal{N}-1)}{E^3} \langle M \rangle c a^{4/3} \left(\frac{1}{4} - \frac{1}{2\mathcal{N}}\right)$$
$$\simeq \frac{9\mathcal{N}^3 c a^{4/3} \langle M \rangle}{4E^3} . \tag{5.6}$$

Note that calculating the mean square force using the Holtsmark distribution, we shall obtain a divergent quantity, so far as the constant c in (5.3-5.6) is formally equal to infinity. This is due to the fact that the Holtsmark distribution predicts too high probabilities for $\vec{\mathcal{E}}^2$ at $|\vec{\mathcal{E}}| \to \infty$, which in turn is connected with the (longlaying) Coulomb character of interaction.

Introducing cutoff * for the forces of the order of [22]

$$|\mathcal{E}_{cutoff}| \sim G M_a / r^2_{cutoff} ,$$
$$r_{cutoff} = 2G(M_a + M_b)/\langle v \rangle^2 , \tag{5.7}$$

where r_{cutoff} is the distance on which the escape velocity (from a star) equals $\langle v \rangle$.

---

* The similar divergency takes place in electrodynamics, which in particular arises when calculating the Lamb shift and is overcome as is well known, by cutoff of contribution of small distances.

we shall get a finite value for $C \simeq 1$.

Now using the above-obtained result for R we can estimate roughly the index of the exponent in (2.12).

Indeed, in the case when $K < 0$ in (2.10 - 2.12), (2.10) can be written as

$$\frac{d^2 |\delta q|^2}{ds^2} > 2K \left| \delta q \wedge \frac{dq}{ds} \right|^2, \qquad (5.8)$$

where

$$K = \min_{(\delta q, \frac{dq}{ds})} \left| K\left(\delta q, \frac{dq}{ds}\right) \right| .$$

The solution (5.8) in turn has the form

$$|\delta q(s)| \gtrless \delta q(0)\, e^{\pm\sqrt{2K}\, s} \quad if \ \frac{d|\delta q|}{ds}\Big/_{s=0} \gtrless 0 \qquad (5.9)$$

Hence the relaxation time is

$$\tau = (2K)^{-1/2}. \qquad (5.10)$$

Now using (2.11) we estimate the mean curvature over two-dimensional directions

$$\bar{K}\left(\delta q, \frac{dq}{ds}\right) \sim \frac{R}{(3N)^2} \qquad (5.11)$$

and from (2.3) we have

$$\left| \frac{dq}{ds} \quad \frac{dq}{ds} \right| = W\left(\frac{dq}{ds}\right)^2 = 1 \qquad (5.12)$$

therefore

$$K \left| \delta q \wedge \frac{dq}{ds} \right|^2 \sim \frac{R}{(3N)^2} \quad |\delta q|^2 \qquad (5.13)$$

Eqs. (2.10), (5.8) and solutions (2.12), (5.9) contain the eigentime ds which is related with the real time by the expression

$$ds = \sqrt{2}\, W dt \,. \tag{5.14}$$

Having in mind (5.13), (5.14), we rewrite (5.8) in the form

$$\frac{d^2}{dt^2}\left|\delta q\right|^2 > \frac{4RW^2}{(3\mathcal{N})^2}\left|\delta q\right|^2 \tag{5.15}$$

Finally we have

$$\left|\delta q(t)\right| \gtrless \left|\delta q(0)\right| e^{\pm 2\sqrt{\frac{RW^2}{\mathcal{N}^2}}\, t} \tag{5.16}$$

and for the relaxation time

$$\tau \simeq \frac{1}{2}\,\frac{3\mathcal{N}}{\sqrt{RW^2}} \,. \tag{5.17}$$

Using the previously obtained expression for the mean curvature radius (5.6) for the relaxation time we shall have

$$\begin{aligned}\tau &\simeq \sqrt{\frac{E}{c\,a^{4/3} < M > \mathcal{N}}} = \\ &= \left(\frac{15}{4}\right)^{2/3} \frac{1}{2\pi\sqrt{2c}}\;\frac{< V >}{G < M > n^{2/3}} \,.\end{aligned} \tag{5.18}$$

In (5.18) $E = \mathcal{N} < M > < V^2 > /2$.

The relaxation time (5.18) normalized on characteristic values of the stellar systems (globular clusters, galaxies) parameters is

$$\tau \simeq 10^8_{yr} \left(\frac{< V >}{10\,\frac{km}{s}}\right)\left(\frac{n}{1\,pc^{-3}}\right)^{-2/3}\left(\frac{< M >}{M_\odot}\right)^{-1} . \tag{5.19}$$

and for the clusters of galaxies 10^{10} - 10^{12} years.

Compare the relaxation time τ with τ_{ch} taking into account the binary encounters only

$$\tau_{ch} = \frac{\sqrt{2}\, V^3}{\pi G^2 \langle M\rangle^2 n \ln(N/2)} . \tag{5.20}$$

The ratio of these relaxation times is

$$\frac{\tau_{ch}}{\tau} \sim \frac{V^2/GM}{n^{1/3}} \frac{1}{\ln N} = \frac{d}{R_*} \frac{1}{\ln N} , \tag{5.21}$$

where R_* is the radius of the star gravitational influence, d is the mean distance between stars.

The expression (5.21) reflects the main physical difference between the relaxation times τ and τ_{ch} which is the following: as we have already mentioned, the relaxation time τ is defined by the curvature of Riemann manifold, the contribution to which is determined simply by the presence of the nearest neighbours at a mean distance d, whereas the contribution into τ_{ch} is determined by binary encounters only, characterized by the effective radius R_*. Since for the real stellar systems $d \gg R_*$, we have

$$\tau \ll \tau_{ch} . \tag{5.22}$$

Note that in the mechanism of collective relaxation discussed above the multiple mutual scatterings of all N bodies including the pair ones are efficiently taken into account. With increasing density, d decreases and approaches R_*, so that the binary encounters become dominating in the relaxation mechanism. That occurs at unreally high stellar densities when $d \sim R_*$. For the globular clusters it will take place at $n \sim 10^{12}$ pc^{-3}.

In the following section the curvature over two-dimensional directions is investigated and the relation (5.11) is derived. Several physical consequences are discussed.

6. The Mean Radius of Curvature Along Two-Dimensional Directions

The previous sections have dealt with the scalar curvature, whereas actually two-dimensional curvature $\mathcal{K}$ (2.11) enters the equation for the geodesic divergency. Below, we give the analysis of the two-dimensional curvature.

Lat us calculate the right-hand side of (2.11) using the expression for the Riemann tensor (4.1). After substitution we obtain

$$
\begin{aligned}
&R_{\alpha\beta\gamma\delta}\,\delta q^{\alpha}\,\dot q^{\beta}\,\delta q^{\gamma}\,\dot q^{\delta} = \\
&= \frac{1}{2W}\Big[2|\dot q W''\delta q||\delta q\dot q| - |\delta q W''\delta q||\dot q\dot q| - |\dot q W''\dot q||\delta q\delta q|\Big] - \\
&- \frac{3}{4W^2}\Big[2|\dot q W'||W'\delta q||\delta q\dot q| - |\delta q W'||\delta q W'||\dot q\dot q| - \\
&- |\dot q W'||W'\dot q||\delta q\,\delta q|\Big] + \frac{1}{4W^2}\Big[|\dot q\delta q|^2 - |\dot q\dot q||\delta q\,\delta q|\Big]\cdot|W'W'|,
\end{aligned}
\tag{6.1}
$$

where the dot over $q\ (\dot q)$ denotes differentiation with respect to S, a dash over $W\ (W')$ differentiation with respect to q, and the bars denote

$$
|\dot q W''\delta q| = \frac{dq^{\alpha}}{ds}\,\frac{\partial^2 W}{\partial q^{\alpha}\partial q^{\beta}}\,\delta q^{\beta}; \qquad |\delta q W'| = \delta q^{\alpha}\,\frac{\partial W}{\partial q^{\alpha}};
$$

$$|W'W'| = \frac{\partial W}{\partial q^{\alpha}} \frac{\partial W}{\partial q^{\beta}} g^{\alpha\beta} \quad .$$

Let us present the linear divergency of the geodesics as the sum of longitudinal and normal components* to the velocity vector

$$\delta q = \delta q_{\perp} + \delta q_{\parallel} \, , \tag{6.2a}$$

$$|\delta q_{\perp} \; \dot{q}| = 0 \, . \tag{6.2b}$$

Then, one can see from (2.8) that the longitudinal component satisfies the trivial equation

$$\frac{D^2 \delta q_{\parallel}^2}{Ds^2} = 0 \tag{6.3}$$

and the normal - the same equation (2.9), so that now in (2.10), (2.11), (5.8) we have $\left|\delta q \wedge \frac{dq}{ds}\right| = |\delta q|^2$. Hence in (6.1) δq may be treated only as normal component, and therefore, using (6.2b) we arrive at

$$\begin{aligned} R_{\alpha\beta\gamma\delta}\, \delta q^{\alpha} \dot{q}^{\beta} \delta q^{\gamma} \dot{q}^{\delta} = & -\frac{1}{2W}\Big[|\delta q W'' \delta q||\dot{q}\dot{q}| \\ & + |\dot{q} W'' \dot{q}||\delta q \delta q|\Big] + \frac{3}{4W^2}\Big[|\delta q W'|^2 |\dot{q}\dot{q}| \\ & + |\dot{q} W'|^2 |\delta q \delta q|\Big] - \frac{1}{4W} |W'W'||\delta q \delta q||\dot{q}\dot{q}| \, . \end{aligned} \tag{6.4}$$

The first two terms in (6.4) are of the same nature for they contain the second derivatives of the potential and correspond to the first term in (4.2) with the Laplacian, while the third, fourth and fifth terms have the first derivatives of the potential and correspond to the second term in (4.2) with the gradient.

Calculate in the explicit form the expressions

$$\left|\delta q W'' \delta q\right| = \sum_{a,b} \sum_{i,k} \partial r_a^i \frac{\partial^2 U}{\partial r_a^i \partial r_b^k} \delta r_k^b = \tag{6.5}$$

$$= G \sum_{a<b} \left\{ \frac{M_a M_b}{r_{ab}^3} \left[\delta \vec{r}_{ab} \, \delta \vec{r}_{ab} - \frac{3(\delta \vec{r}_{ab} \, \vec{r}_{ab})^2}{r_{ab}^2} \right] \right.$$

$$\left. - \frac{4\pi}{3} M_a M_b \left(\delta \vec{r}_{ab} \, \delta \vec{r}_{ab} \right) \delta^{(3)} \left(\vec{r}_{ab} \right) \right\},$$

$$\left|\dot{q} W'' \dot{q}\right| \to \left|\delta q W'' \delta q\right| \quad \text{at} \quad \delta q \to \dot{q},$$

where the second derivative matrix

$$\frac{\partial^2 U}{\partial r_a^i \partial r_b^k} = -G M_a M_b \left\{ \frac{1}{r_{ab}^3} \left(\delta^{ik} - \frac{3 r_{ab}^i \, r_{ab}^k}{r_{ab}^2} \right) \right. \tag{6.5a}$$

$$\left. + \frac{4\pi}{3} \delta_{ik} \delta^{(3)} \left(\vec{r}_{ab} \right) \right\}; \quad a \neq b$$

$$\frac{\partial^2 U}{\partial r_a^i \partial r_b^k} = -G \sum_{\substack{c \\ c \neq a}} M_a M_c \left\{ \frac{1}{r_{ac}} \left(\delta^{ik} - \frac{3 r_{ac}^i \, r_{ac}^k}{r_{ac}^2} \right) - \right. \tag{6.6b}$$

$$\left. - \frac{4\pi}{3} \delta_{ik} \delta^{(3)} \left(\vec{r}_{ac} \right) \right\}; \quad a = b$$

has been used.

Note that the term without $\delta^{(3)}(\vec{r})$ function corresponds to quadrupolar moment of the gravitational system whose trace is zero, so that the total trace of this matrix

$$\Delta U = -\Delta W = 4\pi G \sum_{a \neq b} M_a M_b \delta^{(3)}(\vec{r}_{ab}). \tag{6.7}$$

(see (4.2)-(4.5)). The last three terms can be obtained by means of the first-derivative vector

$$\frac{\partial U}{\partial r_a^i} = \sum_{b, b \neq a} U'_{ab} \frac{\vec{r}_{ab}}{r_{ab}} = \sum_{b, b \neq a} G \frac{M_a M_b}{r_{ab}^3} \vec{r}_{ab}. \tag{6.8}$$

Let us introduce the notations

$$\frac{\partial^2 U}{\partial q^\alpha \partial q^\beta} = A_{\alpha\beta} \quad ; \qquad \frac{\partial U}{\partial q^\alpha} = B_\alpha . \tag{6.9}$$

Consider the case when $r_{ab} \neq 0$ Then the singular part $A_{\alpha\beta}$ turns into zero, while the remaining real part is symmetric and in virtue of (6.6) has a zero trace. Among its eigenvalues λ_α there will be both positive and negative terms. the difference between their numbers being the invariant of this matrix in virtue of the law of inertia. In fact, we are interested in the sign-definiteness of quadratic forms $|\delta q\, A\, \delta q|$ and $|\dot{q} A \dot{q}|$ composed of $A_{\alpha\beta}$ in 3N-dimensional spaces δq^α and $\dot{q}^\alpha$. Since this matrix has the eigenvalues of different signs, the forms $|\delta q\, A\, \delta q|$ and $|\dot{q} A \dot{q}|$ are sign-indefinite, so that the surface where these quadratic forms turn to zero is a hypersurface given by the expression

$$\lambda_1 \delta q_1^2 + \ldots + \lambda_M \delta q_M^2 - \lambda_{M+1} \delta q_{M+1}^2 - \ldots - \lambda_{3N} \delta q_{3N}^2 = 0 \tag{6.10}$$

which at $N = 1$ represents a conical surface (Fig.1). At certain instants of time r_{ab} may become zero, then singular terms will contribute into

(6.4) either. As was mentioned before, such events take place extremely rarely.

To study the sign-definiteness of the last three terms, we shall write down the scalar products in the 3N-dimensional spaces δq^{α} and $\dot{q}^{\alpha}$ in the form

$$\delta q^{\alpha} B_{\alpha} = |\delta q\, \delta q|^{1/2} |B B|^{1/2} \cos \vartheta_{\delta q} , \tag{6.11}$$

$$\dot{q}^{\alpha} B_{\alpha} = |\dot{q}\, \dot{q}|^{1/2} |B B|^{1/2} \cos \vartheta_{\dot{q}} ,$$

where $\vartheta_{\delta q}$ and $\vartheta_{\dot{q}}$ are the angles between vectors δq^{α} , B_{α} and $\dot{q}^{\alpha}$ B_{α} , respectively. Then these terms can be rewritten in the form

$$\frac{3}{4} |\delta q\, \delta q|^{1/2} |B B|^{1/2} \left[\cos^2 \vartheta_{\delta q} + \cos^2 \vartheta_{\dot{q}} - \frac{1}{3} \right] . \tag{6.12}$$

Fig.2 shows the positive and negative regions of the expression from the square brackets in (6.12). The null line is determined by the equation

$$\cos^2 \vartheta_{\delta q} + \cos^2 \vartheta_{\dot{q}} = \frac{1}{3} \tag{6.13}$$

and the maximum and minimum are achieved in the points

$$\vartheta_{\delta q}^{max} = \vartheta_{\dot{q}}^{max} = 0,\ \pi,\ 2\pi . \tag{6.14a}$$

$$\vartheta_{\delta q}^{min} = \vartheta_{\dot{q}}^{min} = \pi/2 ,\ 3\pi/2 . \tag{6.14b}$$

being equal to

$$+\frac{5}{4}|BB||\delta q\delta q||\dot{q}\dot{q}| \quad \text{and} \quad -\frac{1}{4}|BB||\delta q\delta q||\dot{q}\dot{q}| \tag{6.15}$$

respectively.

Thus, during the evolution of the system at different times the form (6.4) can take both positive and negative values.

However with increasing N for spherically symmetric systems, the time the system spends in the region of negative values starts to prevail strongly. In order to make sure of that, let us choose an assembly of systems with spherically symmetric initial velocities and shifts and average (6.4) over them. We shall use here the following relations:

$$\overline{\delta q^{\alpha}\delta q^{\beta}} = \frac{1}{3\mathcal{N}} g^{\alpha\beta} |\delta q\, \delta q| ,$$
$$\overline{\dot{q}^{\alpha}\dot{q}^{\alpha}} = \frac{1}{3\mathcal{N}} g^{\alpha\beta} |\dot{q}\, \dot{q}| \quad . \tag{6.16}$$

As a result of such averaging, the first two terms in (6.4) transform into expression

$$-\frac{\Delta W}{3\mathcal{N}W^2} |\dot{q}\, \dot{q}||\delta q\, \delta q| \tag{6.17}$$

which is zero at $r_{ab} \neq 0$.

The last three terms in (6.4) after averaging are equal to

$$\left(\frac{1}{2\mathcal{N}} - \frac{1}{4}\right) \frac{(\nabla W)^2}{W^3} |\delta q\, \delta q||\dot{q}\, \dot{q}| , \tag{6.18}$$

One can see that with increasing N the system most of the time spends near the negative values (6.14b) equal to $-\frac{1}{4}B^2$.

After averaging the expression (6.4) looks like

$$\overline{R_{\alpha\beta\gamma\delta}\,\delta q^{\alpha}\dot{q}^{\beta}\delta q^{\gamma}\dot{q}^{\delta}} =$$

$$= \left\{-\frac{1}{3\mathcal{N}}\,\frac{\Delta W}{W^2} + \left(\frac{1}{2\mathcal{N}} - \frac{1}{4}\right)\frac{(\nabla W)^2}{W^3}\right\}\left|\delta q\;\delta q\right| = \tag{6.19}$$

$$= \frac{R}{3\mathcal{N}(3\mathcal{N}-1)}\cdot\left|\delta q\;\delta q\right| = K\left|\delta q\;\delta q\right|,$$

wherein we have used the relations (4.2), (6.7), (5.12). Thus, the validity of the relation (5.11) is shown.

Thus, generally speaking, the dynamics of the system is not determined by the scalar curvature R only, however, as is seen from (6.19), the averaged curvature over two-dimensional directions K is proportional to scalar curvature and turns out negative in virtue of verified analysis.

At the same time, the dynamics of the non-spherical systems is determined by two-dimensional curvature K which may be both positive and negative. This significant fact is responsible for that rich variety of configurations of stellar systems that are observed in the Universe! In particular, the presence of regions with positive curvature admits the existence of the systems with a more regular structure (cf. [25, 26]) than globular clusters and elliptical galaxies, namely spiral galaxies of different classes.

Thus, the initial data of stellar systems, notwithstanding their identical dynamics, determine essentially the subsequent evolution of the system. At spherical distribution of the initial data, as it was shown above, the system possesses exponential instability and strong statistical properties: those systems are globular clusters and elliptical galaxies. On the other hand, in the presence of the initial rotational moment the system most of the time spends in the regions with positive two-dimensional curvature and hence has a more regular structure, as, for example, spiral galaxies have. Hence it follows that the elliptical and spiral galaxies must have different origin, which is just confirmed by recent investigations.

As is well known, according to the KAM theory [27] , the phase space of the non-integrable system contains both the regions of regular motion (the torus winding) and those of irregular motion. From this point of view, the elliptical galaxies are in the region of irregular motion, and the spiral ones - in the region of more regular one, close to some integrable problem, the small parameter being inverse proportional to rotational moment.

7. Discussion and Conclusion

The statistical properties of the stellar systems from a position of the ergodic theory have been investigated in the present work. The initial point was to reduce the N-body problem to the investigation of the behaviour of the geodesic flux on the Riemann manifold. The exponential divergency of geodesics was shown for the stellar systems with the exponent index determined by curvature of this manifold. According to the definition (Sec.4), we have taken the index of this exponent as the relaxation time. A question that naturally arises here is how the determined time of relaxation is connected with the time of arriving of the system to equilibrium state. This question was discussed in detail in the remarkable works of N.S.Krylov [12] on the foundations of statistical physics. Below, we shall dwell briefly on this question.

The question is what is the rate the primary cell of phase space will cover uniformly the energy hypersurface H = const. In the ergodic theory 10 it is shown that in the mixing systems the primary cell complicates its shape so much (preserving its volume and bond), that at $t \to \infty$ covers uniformly the hypersurface H = const.* In this sense, precisely in the systems with mixing, any non-equilibrium distribution tends to the equilibrium one during infinite time. However, if requiring that the mixing occurs with the prescribed accuracy ε connected with the accuracy of physical measurement, then this time τ_ε will be finite.

The systems with exponential instability, i.e. those with K-mixing, tend to equilibrium state with the exponential rate $e^{s/\tau}$. Therefore, if we adopt a certain accuracy of the equilibrium state arrived, then the relaxation time τ_ε will be expressed via a characteristic time τ :

$$\tau_\varepsilon = \mathcal{N}(\varepsilon)\tau ,$$

where $\mathcal{N}(\varepsilon)$ is the number that depends only on ε .

Compare this definition of the relaxation time with that of Chandrasekhar [1] . The initial point of our definition is the trajectory instability which implies that the change of the initial data on $\delta q\,(0)$ is carried out with exponential rate: $\delta q(s) \sim \delta q(0)e^{s/\tau}$. Thus, e.g., if we consider the binary encounters of the stars with the initial scattering angles φ_{in}, and $\varphi_{in} + \Delta\varphi_{in}$, then the exponential instability means that

$$\Delta\varphi_{out} = K\Delta\varphi_{in}$$

---

* The ergodic systems do not possess this property.

in the process of a single act of encounter. And after n encounters the angle divergency will be

$$\Delta\varphi_{out}^{(n)} = K^n \Delta\varphi_{in} = \Delta\varphi_{in} e^{n \ln K}$$

so that

$$\tau = 1/\ln K$$

As for Chandrasekhar [1] , he considers the differnce

$$\Delta\varphi = \varphi_{out} - \varphi_{in} ,$$

and the relaxation time the time during which

$$\Delta\varphi^{(n)} \sim \pi/2$$

owing to the binary encounters, i.e. when the change of the angle of a single star will become of the order of $\pi/2$.

Let us formulate the main results of the study.

1. We have shown that an exponential instability which leads to equilibrium state exists in the stellar systems.

2. The expression for the curvature tensor determining the declination of geodesics is analyzed. The regions of positivity and negativity of the two-dimensional curvature are realized and its role in the evolution of the stellar systems is discussed.

3. The magnitude of the relaxation time for the real systems (globular clusters and galaxies) is calculated. The time obtained is compared with the Chandrasekhar relaxation time and their physical differences are discussed.

In conclusion we mention several additional astrophysical consequences. As is well known, if the velocity distribution of stars is Maxwellian, some part of the stars will evaporate [24] . So far as the time τ is essentially less than the Chandrasekhar relaxation time, the role of evaporation process in the evolution of the stellar system increases sharply. The obtained results can serve as a foundation to the hypothesis of the local equilibrium of the stellar systems [28, 29].

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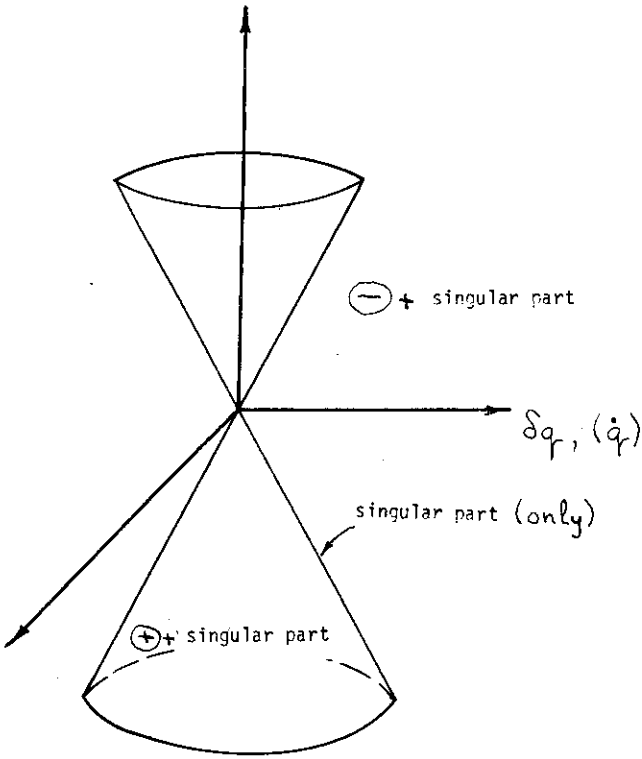

Fig. 1. Schematic representation of the regions of positive and negative sign of the first two terms from Eq.(6.4).

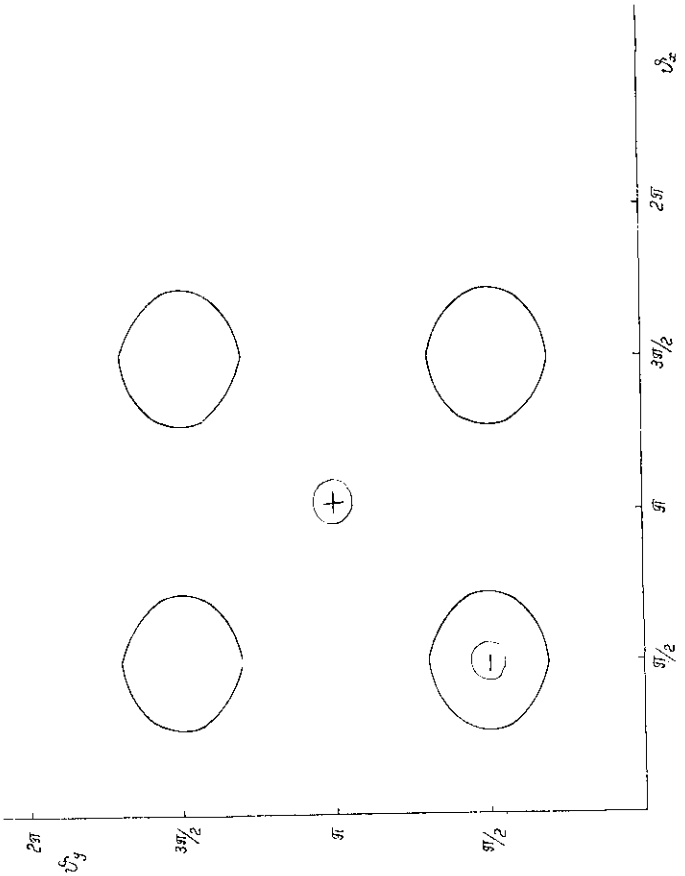

Fig. 2. Regions of positive and negative sign of the last three terms from Eq.(6.4).

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