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ЦЕНТРАЛЬНЫЙ НАУЧНО-ИССЛЕДОВАТЕЛЬСКИЙ ИНСТИТУТ
ИНФОРМАЦИИ И ТЕХНИКО-ЭКОНОМИЧЕСКИХ ИССЛЕДОВАНИЙ
ПО АТОМНОЙ НАУКЕ И ТЕХНИКЕ

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SIGN FACTOR OF THREE-DIMENSIONAL ISING MODEL
AND THE QUANTUM SUPERSTRING

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Yerevan 1984
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The crucial role played by gauge symmetry in the physics of strong interactions is generally accepted today. The elementary excitations of gauge theories seem to have the string nature. It means that in quantum case one will face the problem of summing over the surfaces. An essential step in this direction was made recently by A.M.Polyakov [1] .

The simplest model of fluctuating surfaces is the free bosonic string. It is described by the sum over the surfaces (selfintersecting and non-selfintersecting) with the statistical weight depending only on the area of the surface. To get the description of the fermionic string, let us recall the difference between bosonic and fermionic particles in the functional integral picture. The study of two-dimensional Ising model (2d IM) has shown that the propagation of the two-dimensional fermion can be described as a random walk, but the path integral must include only non-selfintersecting paths. This is achieved by providing each statistical weight in the functional integral by the sign factor (Kac-Ward factor), ensuring the cancellation of the contributions of selfintersecting trajectories.

Proceeding by analogy it is natural to define the theory of a free fermionic string in three dimensions by the functional integral over non-selfintersecting surfaces with the weight $e^{-\alpha A}$, A being the area of the surface. The free energy of so defined string will coincide with the

free energy of three-dimensional Ising gauge model (3d IM) near the critical point. The latter is known to be the sum over non-selfintersecting surfaces on the regular lattice. The main problem in this approach is the definition of the sign factor ensuring the necessary cancellations and its construction in terms of the fermionic variables on the surface analogous to the construction found in the fermionic particle case [2, 3].

The present paper consists of 2 sections. In sec.1 the expression for the sign factor and the general mathematical considerations leading to it are given in some details. We also give the expression for the sign factor on a regular lattice proposed by Polyakov and Dotsenko [2]. It is shown that our sign factor is the generalization of Polyakov-Dotsenko one. The generalization allows one to pass from the surfaces on a regular lattice to arbitrary piecewise smooth surfaces which are essential near the critical point.

In sec.2 we present some calculations showing the possible equivalence of the continuum limit of 3d IM to the fermionic part of the supersymmetric string defined in the work [4]. Making some plausible assumptions about the calculation of the functional integral over the anticommuting variables we show that the free energy of the superstring is equal to the sum of free energies of bosonic and fermionic strings written in the form of the functional integral over the surfaces $\vec{X}(\xi)$. The fermionic part of the free energy contains, besides the weight factor $e^{-\alpha A}$, the sign factor defined in sec.1. The analogous result is known to hold for the free two-dimensional superparticle, the fermionic component of it defining the continuum limit of 2d IM [3].

1. The Sign Factor of 3d IM.

The partition function of 3d IM can be presented in the form

$$Z = (ch\beta)^N \sum (th\beta)^A, \quad (1)$$

where $\beta = \frac{J}{KT}$, N is the number of the plaquettes of three-dimensional regular lattice, and the summation is carried out over the closed surfaces on the lattice, A being the area of the surface. The sum (1) includes the surfaces of the type shown in Fig.1a. It is desirable however to deal only with the surfaces which can be parametrized by the function $\vec{X}(\xi_1, \xi_2)$. This would allow one to write the partition function near the critical point as a functional integral over $\vec{X}(\xi_1, \xi_2)$. It is known [5] that this can be achieved by providing each statistical weight in (1) by a sign factor $(-1)^\ell$, where ℓ is the total length of the lines of selfintersections of the surfaces (Fig.1)

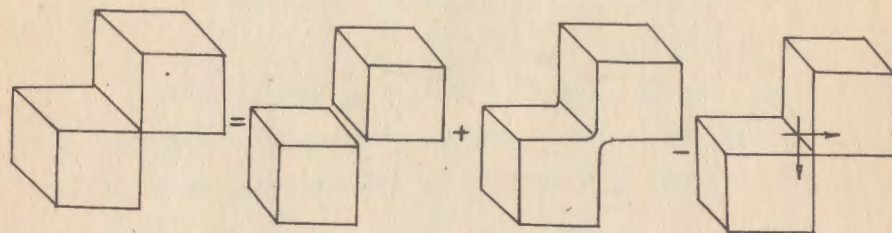


Fig. 1.

The partition function thus takes the form

$$Z = (ch\beta)^N \sum_{\text{over two-dimensional surfaces}} (th\beta)^A (-1)^\ell. \quad (2)$$

One may say that the contributions of the surfaces of the types (c) and (d) cancel each other, and what is left is the sum over non-selfintersecting multiconnected surfaces.

Now the question arises how to write the sign factor $(-1)^\ell$ in terms of the differential-geometric characteristics of the surfaces. In their beautiful work [2] Polyakov and Dotsenko suggest the following program. Let's cross the surface on a regular lattice by the normal planes, crossing every link of the lattice once. The planes will intersect the surface by some closed lines (C_1, \dots, C_6 in Fig.2)

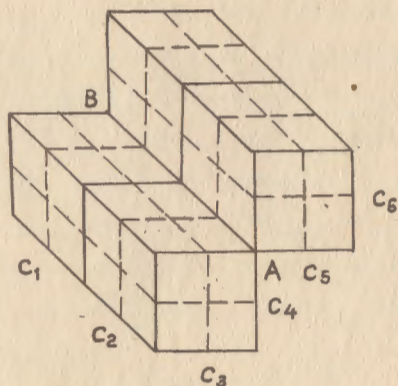


Fig. 2

If the surface has the line of selfintersection (AB in Fig.2), then on the planes crossing that line one will have the selfintersecting contours, while the other contours will have no selfintersections. If the total length of lines of selfintersections is ℓ , then the contours will have ℓ points of selfintersections. Now providing each contour by the Kac-Ward factor of 2d IM and taking the product of those factors one will arrive at the necessary sign factor $(-1)^\ell$ of 3d IM. On the other hand, the 2d IM near the critical point is known to be equivalent to the free massless fermion theory and, as was shown by Polyakov and Dotsenko [2], the partition function of 2d IM can be written near the critical point as a path integral with the action proposed by Casalbuony [6] and Berezin and Marinov [7] for the fermionic particle:

$$Z = \int \mathcal{D}\vec{x}(\tau) \mathcal{D}\vec{\xi}(\tau) \mathcal{D}x(\tau) e^{-S}, \quad (3)$$

$$S = \int d\tau \left(\frac{i}{2} \vec{\xi} \dot{\vec{\xi}} + \frac{i}{2} x \dot{x} + x \vec{\xi} \dot{\vec{x}} \right).$$

These considerations show that it is natural to expect that the 3d IM near the critical point is described by Neveu-Schwarz fermionic string action [8, 9], which is the generalization of the action (3) in the case of strings.

The geometrical meaning of Kac-Ward factor is well known. It is the exponent of the tangent vector rotation angle along the contour

$$(-1)^m = -e^{\frac{i}{2} \oint \vec{e} d\vec{e}} \quad (4)$$

In three-dimensional space this quantity may also be written as

$$(-1)^m = -\frac{1}{2} \text{tr} e^{\frac{i}{4} \oint (\vec{e}_d \vec{\sigma}) (\vec{n} \vec{\sigma})}, \quad (5)$$

where m is the number of selfintersections of the contour, $\vec{\sigma}$ are Pauli matrices, \vec{n} is the unit vector normal to the plane of the contour.

The described construction of the sign factor is strictly defined only for the surfaces on the regular lattice. However, near the critical point, where the system is scale-invariant and one must tend the lattice spacing to 0, the arbitrary piecewise smooth surfaces become important. To get the expression for the sign factor for such surfaces, let us consider the regular homotopical classes of immersions of two-dimensional manifolds into three-dimensional space.

The lines of selfintersection of the surfaces may be either open (i.e. having the end points, Fig.1d) or closed. In the former case one has piecewise smooth immersed surface with singularities on the ends of the lines of selfintersection, while in the latter case the immersed piecewise smooth surface with no singularities. The regular immersions of two-dimensional surfaces into the three-dimensional Euclidean space were investigated by Whitney [10], Smale [11], and Hirsch [12]. It follows from their works that the classification of the immersed surfaces with respect to the regular homotopy is connected with the homotopical groups of the mappings of S^1 (the contours) into the group $SO(3)$, $\mathcal{P}T_1(SO(3))$.

In Fig.3 two examples of the immersion of the torus into three-dimensional space are given. They belong to different homotopical classes.

In Fig.3a the contour C crosses the line of selfintersection, while in Fig.3b there is no selfintersection at all.

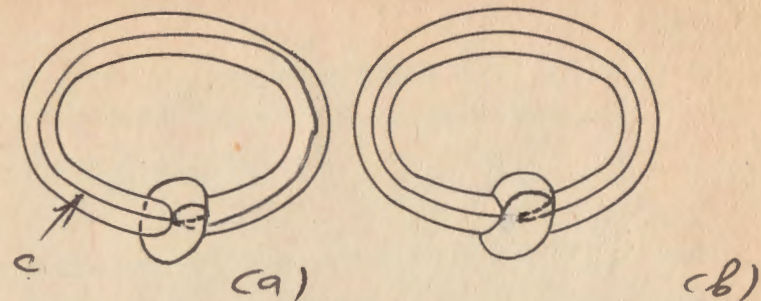


Fig. 3

To write down the element of $\mathcal{P}T_1(SO(3))$ in terms of the characteristics of the contour C , consider in each point of the contour three orthogonal vectors \vec{e}_α ($\alpha = 1, 2, 3$). We take \vec{e}_1 to be tangent to the contour, \vec{e}_2 - tangent to the surface and normal to the contour, \vec{e}_3 is normal to the surface. Now consider the matrix

$$\Omega = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \in SO(3) \quad (6)$$

and consider the quantity

$$\Phi(C) = \frac{1}{2} \text{tr} P \exp \oint_C \Omega^{-1} d\Omega, \quad (7)$$

where P is the ordered product operator. In (7) we take all the generators of $SO(3)$ in spinor representation. In this case $\Phi(C)$ may be written in the form

$$\Phi(C) = \frac{1}{2} \text{tr} P \exp \left\{ \frac{i}{4} \oint_C \hat{e}_\alpha d\hat{e}_\alpha \right\} \in \mathbb{Z}_2, \quad (8)$$

where

$$\hat{e}_\alpha = \vec{e}_\alpha \vec{\sigma}$$

Indeed,

$$\begin{aligned} \Omega^{-1} d \Omega &= (\vec{e}_1, \vec{e}_2, \vec{e}_3) d \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} = \\ &= \frac{1}{2} \varepsilon_{\alpha\beta\gamma} \vec{e}_\alpha d \vec{e}_\beta \Gamma_\gamma = i \Gamma_\alpha \Gamma_\beta \vec{e}_\alpha d \vec{e}_\beta \end{aligned} \quad (9)$$

where Γ_α are the $SO(3)$ generators in adjoint representation. Making the substitution $\Gamma_\alpha \rightarrow \frac{\sigma_\alpha}{2}$ into $\Omega^{-1} d \Omega$ (the element of the algebra) one arrives at the formula (8).

It is easy to see that $\Phi(c)$ remains unchanged when the contour is moving within the fixed homotopical class of the contours. Indeed, the curvature of the connection $\Omega^{-1} d \Omega$ is equal to 0. But if the contour crosses a singular point in which Ω is not defined as an element of $SO(3)$ (e.g. the point A in Fig.4), then $\Phi(c)$ changes.

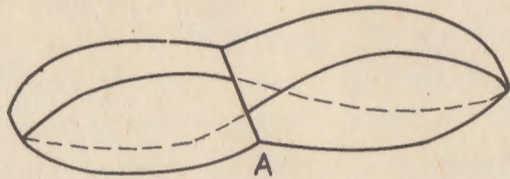


Fig. 4

It means that $\Phi(c)$ depends only on the number of singular points enclosed by the contour C . Below we will show that

$$\Phi(c) = -(-1)^m,$$

where m is the number of times the contour C crosses the line of selfintersection. For example, for the contour in Fig.3a $\Phi(c) = 1$, for the contour in Fig.3b $\Phi(c) = -1$, for the contour in Fig.4 $\Phi(c) = -1$.

Let us consider the geometrical meaning of $\Phi(c)$. For this purpose approximate the contour by the broken line and write the discrete version of (7). In adjoint representation one has (see Fig.5)

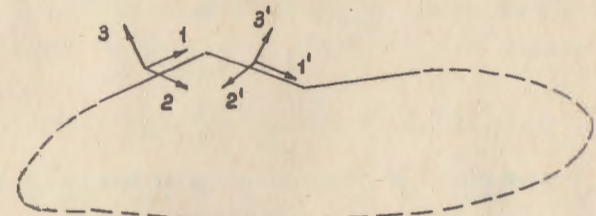


Fig. 5

$$\begin{aligned} \Phi(c) &= \frac{1}{2} \text{tr} P \prod_i \exp [\Omega_i^{-1} (\Omega_{i+1} - \Omega_i)] = \\ &= \frac{1}{2} \text{tr} P \prod_i \Omega_i^{-1} \Omega_{i+1}. \end{aligned} \quad (10)$$

The matrix $\Omega_i^{-1} \Omega_{i+1}$ rotates the vectors \vec{e}_α^i in the point i into the vectors \vec{e}_α^{i+1} .

$$\vec{e}_\alpha^i \Omega_i^{-1} \Omega_{i+1} = \vec{e}_\alpha^{i+1}. \quad (11)$$

In spinor representation the normalized rotation operator has the form

$$S_{i,i+1} = \frac{1 + \hat{e}_\alpha^i \hat{e}_\alpha^{i+1}}{2\sqrt{1 + \vec{e}_\alpha^i \vec{e}_\alpha^{i+1}}} \quad (12)$$

Using the formula

$$\hat{e}_\alpha = \frac{i}{2} \varepsilon_{\alpha\beta\gamma} \hat{e}_\beta \hat{e}_\gamma \quad (13)$$

it is easy to check that

$$\hat{e}_\alpha^i S_{i,i+1} = S_{i,i+1} \hat{e}_\alpha^{i+1}. \quad (14)$$

The $\phi(c)$ is the trace of the ordered product of the rotation matrices along the contour

$$\phi(c) = \frac{1}{2} \text{tr} \prod_i S_{i,i+1}. \quad (15)$$

After the rotation around the closed contour one arrives back at the initial vectors, so $\prod_i S_{i,i+1}$ represents the rotation by the angle $2\pi K$ ($K = 0, \pm 1, \pm 2, \dots$) around some axis \vec{n}

$$\prod_i S_{i,i+1} = e^{i 2\pi K \frac{\vec{n} \cdot \vec{\sigma}}{2}} = \cos \pi K \quad (16)$$

and

$$\phi(c) = \pm 1 \in \mathbb{Z}_2. \quad (17)$$

Now we shall prove the important property of multiplicativity of the factors $\phi(c)$, which follows from the Stokes theorem. The contours C_1 and C_2 in Fig. 6 are obtained by the arbitrary division of the contour C into 2 parts. The points A and B are infinitely close to each other

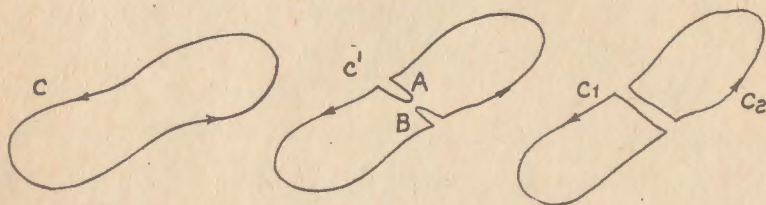


Fig. 6

One can readily prove that

$$\phi(c) = -\phi(c_1) \phi(c_2). \quad (18)$$

Note, first, that since the variations not crossing the singular points do not change the $\phi(c)$, $\phi(c) = \phi(c')$. On the other hand,

$$\phi(c') = \frac{1}{2} \text{tr} \{ \bar{\Phi}(\bar{C}_{AB}) O_B \bar{\Phi}(\bar{C}_{BA}) O_A \}, \quad (19)$$

where $\bar{\Phi}(\bar{C}_{AB})$ and $\bar{\Phi}(\bar{C}_{BA})$ are the products of the rotation matrices from point A to B along the loops shown in Fig. 6b, and O_A and O_B are the operators of rotation by the angle π around the normal to the points A and B . Since on the ends of \bar{C}_{AB} and \bar{C}_{BA} the vectors \vec{e}_α coincide, one has, using (16)

$$\bar{\Phi}(\bar{C}_{AB}) = \phi(c_1) 1$$

$$\bar{\Phi}(\bar{C}_{BA}) = \phi(c_2) 1$$

and

$$\phi(c) = \phi(c_1) \phi(c_2) \frac{1}{2} \text{tr} (O_B O_A) = \quad (21)$$

$$= -\phi(c_1) \phi(c_2),$$

providing

$$i \frac{\pi}{2} \vec{n} \cdot \vec{\sigma}$$

$$O_A = O_B = e$$

Evidently, defining the quantity

$$\chi(c) = -\phi(c) \quad (22)$$

one will have

$$\chi(c) = \chi(c_1) \chi(c_2). \quad (23)$$

Let us now consider the value of $\Phi(c)$ for different contours. The two-dimensional surfaces may be

- i) imbedded into three-dimensional space (no selfintersections)
- ii) immersed (a closed line of selfintersection)
- iii) immersed with the singularities in some points (on the ends of the open lines of selfintersection).

The contours on two-dimensional surfaces may be either homological to zero (the ones that can be deformed into the points) or not (those enclosing the hole or the handle, see Fig.7).

If the contour is homological to 0 and has no singularities of Ω inside, then, clearly

$$\chi(c) = -\Phi(c) = 1.$$

The same result holds if the contour is not homological to 0 and does not pass through the line of selfintersection of the surface (Fig.7)

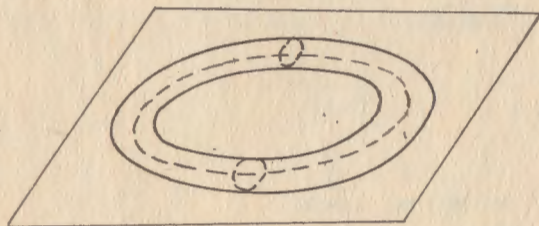


Fig. 7

Consider now the immersed torus (Fig.3a). When passing through the line of selfintersection, the vectors undergo the additional rotation by 2π , so

$$\chi(c) = -\Phi(c) = -1.$$

If one has the contour enclosing the end of the line of selfintersection (Fig.4), then one can smoothly deform it to the contours of type C_1 and C_2 in Fig.2, and our sign factor becomes the usual Kac-Murd factor, yielding

$$\chi(c) = -\Phi(c) = -1.$$

The general statement is that $\chi(c) = -1$ for the contours passing once through the line of selfintersection of the surface, and $\chi(c) = 1$ for the contours not passing through it.

In Appendix some explicit calculations of the value of $\chi(c)$ are presented.

The expression (8) for the sign factor of the contours was originally derived from general mathematical considerations in Ref. [13].

Now our aim is to use the sign factor (7) to construct the sign factor of the surface $(-1)^{\ell}$, where ℓ is the total length of the lines of self-intersections of the surface. For this purpose one must cover the whole surface by the contours passing once through each link of the surface. This covering may be done in different ways giving the different numbers of the contours. In particular, it is possible to have only one contour crossing all the links of the surface, say, in the middle points. The result will not depend on the particular type of covering provided the sign factor (7) has the properties of invariance and multiplicativity.

Any type of the covering with the contours passing through the middles of the links may be obtained using the following procedure. Draw two parallel

lines connecting the middles of the neighbouring links of each plaquette of the surface (see Fig.8a).

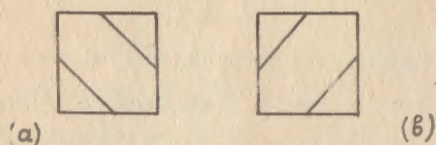


Fig. 8

The surface will get covered by some number of non-intersecting contours. Any other type of covering may now be obtained by switching the links on some plaquettes in the way shown in Fig.8b.

A simple example is given in Fig.9. The contours A and B in Fig.9a are united in Fig.9b to give the contour C.

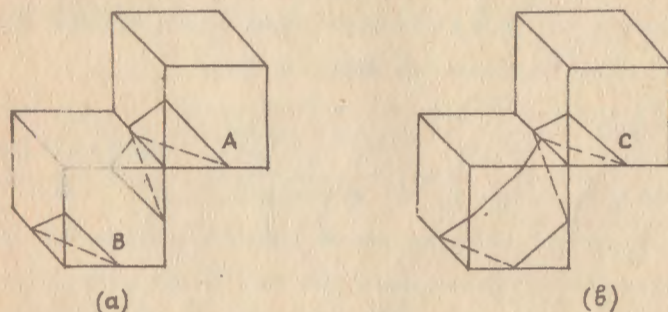


Fig. 9

Our last step is to define the sign factor of the surface Φ as the product of the sign factors of all contours covering the surface

$$\Phi = \prod \Phi(c_i) \quad (24)$$

It is easy to see that it satisfies the necessary equation

$$\Phi = (-1)^g \quad (25)$$

2. The Quantum Supersymmetric String

The idea that the continuum limit of 3d IM can be described in terms of the surfaces with some fermionic structure on them belongs to Polyakov. In his paper [1] he claims that the resulting theory is a properly quantized Neveu-Schwarz string model.

Here we shall consider the free string in three-dimensional superspace. The action for it was proposed in [4]. We shall show that at least in some particular scheme of regularization of the functional integral the fermionic component of it describes the continuum limit of 3d IM.

Consider the superspace $z^M = (\vec{x}, \theta)$, where x^i ($i=1, 2, 3$) are the coordinates of three-dimensional Euclidean space and θ is two-component Majorana spinor. The action of the superstring is given by the expression

$$S = m^2 \int d^2\xi \sqrt{\det(\partial_a x^i - i\bar{\theta}\sigma^i\partial_a\theta)(\partial_b x^i - i\bar{\theta}\sigma^i\partial_b\theta)}, \quad (26)$$

which is invariant under the reparametrization and supersymmetry transformations. In (26) $\bar{\theta} = -i\theta^T\sigma_2$, and $z^M(\xi) = (x^i(\xi), \theta(\xi))$

defines the world surface of the string. The action (26) is the direct supersymmetrical generalization of Nambu-Goto string action.

Our aim is to calculate the free energy of the superstring

$$-F = \int \mathcal{D}\vec{x}(\xi) \mathcal{D}\theta(\xi) e^{-S} \quad (27)$$

Here the integration is carried out over the connected closed surfaces. We choose the conformal gauge

$$\partial_\alpha x^i \partial_\beta x^i = \rho(\xi) \delta_{\alpha\beta} \quad (28)$$

and expand the square root in the action in Taylor series in variable $\bar{\theta} \hat{\epsilon}^\alpha \partial_\alpha \theta$

$$S = m^2 \int d^2 \xi \rho - im \int d^2 \xi \sqrt{\rho} \bar{\theta} \hat{\epsilon}_\alpha \partial_\alpha \theta - \int d^2 \xi \bar{\theta} \hat{\epsilon}_\alpha \partial_\alpha \theta \bar{\theta} \hat{\epsilon}_\beta \partial_\beta \theta \epsilon_{\alpha\beta} - \frac{1}{2} \int d^2 \xi (\bar{\theta} \hat{\epsilon}_\alpha \partial_\alpha \theta)^2. \quad (29)$$

Here $\bar{e}_\alpha = \frac{\partial_\alpha \vec{x}}{\sqrt{\rho}}$ ($\alpha = 1, 2$) are the unit tangent vectors of the surface, $e_3^i = n^i = \epsilon^{ijk} e_j^k e_k^i$ is the normal vector. We also have passed to dimensionless θ by the rule $\theta \rightarrow \frac{1}{\sqrt{m}} \theta$. The parameter space is the disk with some boundary conditions defining the topology of the surface.

We now want to compute the integral over θ 's. For this purpose, we discretize the parameter space in the following way. First, we divide it into squares of size 2ϵ , and then, we place the variables θ in the points $m = (m_1 \epsilon, m_2 \epsilon)$ in the way shown in Fig. 10.

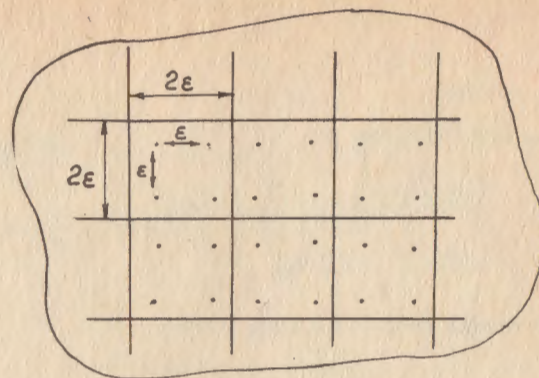


Fig. 10

Then we make the substitutions in the action

$$\partial_\alpha \theta \rightarrow \frac{1}{\epsilon} (\theta_{m+\hat{\alpha}} - \theta_m), \quad (30)$$

$$\bar{\theta} \rightarrow \frac{1}{\epsilon} (\bar{\theta}_{m+\hat{\alpha}} + \bar{\theta}_m),$$

$$\bar{\theta} \epsilon^i \partial_\alpha \theta \rightarrow \frac{1}{\epsilon} \bar{\theta}_m \epsilon^i \theta_{m+\hat{\alpha}}, \quad (31)$$

$$\int d^2 \xi \rightarrow \epsilon^2 \sum_m. \quad (32)$$

and define the measure

$$\mathcal{D}\theta = \prod_m i d^2 \theta_m, \quad \int d^2 \theta_m \theta_m^\alpha \bar{\theta}_m^\beta = i \delta^{\alpha\beta} \quad (33)$$

The θ -dependent terms in the action take the form

$$\sum_m \left\{ -im \epsilon \sqrt{\rho} \bar{\theta}_m \hat{\epsilon}_\alpha \theta_{m+\hat{\alpha}} - \frac{1}{2} \bar{\theta}_m \hat{\epsilon}_\alpha \theta_{m+\hat{\alpha}} \bar{\theta}_m \hat{\epsilon}_\beta \theta_{m+\hat{\beta}} \epsilon_{\alpha\beta} - \frac{1}{2} (\bar{\theta}_m \hat{n} \theta_{m+\hat{\alpha}})^2 \right\}. \quad (34)$$

In the limit $\epsilon \rightarrow 0$ this expression goes into the corresponding part of

expression (29).

Using the equations

$$\begin{aligned} \bar{\theta}_m \bar{\theta}_{m+\hat{a}} &= -\bar{\theta}_{m+\hat{a}} \bar{\theta}_m, \\ \theta_m^\alpha \bar{\theta}_m^\beta &= \theta_m^\alpha \theta_m^\beta \delta^{\alpha\beta} \end{aligned} \quad (35)$$

one can rewrite the last two terms in (34):

$$\begin{aligned} &\theta_m^1 \theta_m^2 \bar{\theta}_{m+\hat{z}} \hat{e}_\beta \hat{e}_\alpha \varepsilon_{\alpha\beta} \theta_{m+\hat{y}} - \\ &-\frac{1}{2} \text{tr} \hat{n}^2 \theta_m^1 \theta_m^2 \theta_{m+\hat{a}}^1 \theta_{m+\hat{a}}^2. \end{aligned}$$

Finally, we discretize the product $\hat{e}_\alpha \hat{e}_\beta$

$$\begin{aligned} \hat{e}_\alpha \hat{e}_\beta &\rightarrow \frac{1}{2} \left\{ \frac{\hat{e}_{\alpha, m+\hat{y}} + \hat{e}_{\alpha, m+\hat{z}}}{2} \hat{e}_{\beta, m+\hat{y}} + \right. \\ &\left. + \hat{e}_{\alpha, m+\hat{z}} \frac{\hat{e}_{\beta, m+\hat{y}} + \hat{e}_{\beta, m+\hat{z}}}{2} \right\}. \end{aligned}$$

The method is to write the expression $\frac{\hat{e}_{\alpha, m+\hat{y}} + \hat{e}_{\alpha, m+\hat{z}}}{2}$ instead of each vector \hat{e}_α and then to omit the terms violating the order of arrangement of θ s. For instance, if one has $\bar{\theta}_{m+\hat{z}}$ in the left of $\theta_{m+\hat{y}}$, then $\hat{e}_{m+\hat{z}}$ must also be kept in the left of $\hat{e}_{m+\hat{y}}$. Such a choice is made for convenience.

Hence the free energy takes the form

$$\begin{aligned} -F &= \int \mathcal{D}\vec{x} e^{-m^2 A} \int \prod_m d^2 \theta_m \exp \left\{ \sum_m i m \varepsilon \sqrt{\rho} \right. \\ &\left. \bar{\theta}_m \frac{\hat{e}_{\alpha, m} + \hat{e}_{\alpha, m+\hat{a}}}{2} \theta_{m+\hat{a}} \right\} \end{aligned}$$

$$\begin{aligned} &+ \frac{i}{4} \theta_m^1 \theta_m^2 \bar{\theta}_{m+\hat{z}} \Omega_{m+\hat{z}, m+\hat{y}}^0 \theta_{m+\hat{y}} + \\ &+ \theta_m^1 \theta_m^2 (\theta_{m+\hat{y}}^1 \theta_{m+\hat{z}}^2 + \theta_{m+\hat{z}}^1 \theta_{m+\hat{y}}^2), \end{aligned} \quad (36)$$

where

$$\Omega_{m, n}^0 = \hat{n}_m + \hat{n}_n - i \varepsilon_{\alpha\beta} \hat{e}_{\alpha, m} \hat{e}_{\beta, n}. \quad (37)$$

Now it is easy to integrate over the θ s. Let us expand the exponent in Taylor series. The nonvanishing contributions will come from the terms having two θ s in each point of the parameter space. The first term in the exponent in (36) is of the order of ε , so we assume that in the limit $\varepsilon \rightarrow 0$ it makes no contribution into the integral. For the second and third terms we introduce the following diagrams

$$\begin{aligned} &\theta_m^1 \theta_m^2 \bar{\theta}_{m+\hat{z}} \Omega_{m+\hat{z}, m-\hat{y}}^0 \theta_{m-\hat{y}} \quad \begin{array}{c} m-\hat{y} \\ \diagdown \\ m \quad m+\hat{z} \end{array} \\ &\theta_m^1 \theta_m^2 \bar{\theta}_{m-\hat{z}} \Omega_{m-\hat{z}, m+\hat{y}}^0 \theta_{m+\hat{y}} \quad \begin{array}{c} m-\hat{z} \\ \diagdown \\ m \quad m+\hat{y} \end{array} \\ &\theta_m^1 \theta_m^2 \theta_{m+\hat{y}}^1 \theta_{m+\hat{z}}^2 \quad \begin{array}{c} \uparrow \\ m \\ \uparrow \\ m+\hat{y} \end{array} \\ &\theta_m^1 \theta_m^2 \theta_{m+\hat{z}}^1 \theta_{m+\hat{y}}^2 \quad \begin{array}{c} m \quad m+\hat{z} \\ \leftarrow \quad \rightarrow \end{array} \end{aligned} \quad (38)$$

Here the θ 's are placed on the ends of the solid lines and the dotted line corresponds to the matrix Ω° . The arrows show the order of indices.

The integrand may be represented now in the following form

$$\prod_m \left\{ 1 + \frac{i}{4} m \triangle + \frac{i}{4} \nabla + m \parallel + \overline{\overline{m}} \right\} \quad (39)$$

The nonvanishing contribution comes only from the terms having two solid lines in each point m . They are described by the diagram shown in Fig. 11a for the second and third terms in (39) and in Fig. 11b for the fourth and fifth terms.

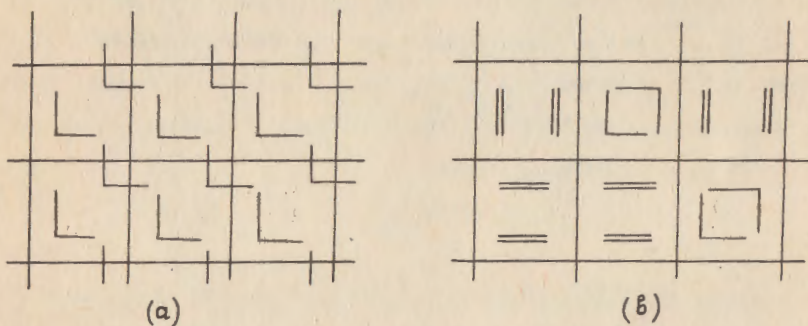


Fig. 11

One may always choose the parametrization and the boundary conditions in such a way that the diagram of the type of Fig. 11a will consist of only one closed contour. Then the free energy will become the sum of two terms corresponding to diagrams of Figs. 11a and 11b.

The first term is given by the expression

$$2 \left(-\frac{1}{16}\right)^N \text{tr} P \prod_c \Omega_{n, n+\epsilon}^\circ, \quad (40)$$

where $N = \frac{A}{4\epsilon^2}$ is the number of the plaquettes of the surface and the product of matrices is taken along the contour. It is easy to notice that the product (40) is proportional to 2ϕ , where ϕ is the sign factor of the surface (24). Really, there is the following relation between Ω° and

$$\begin{aligned} \Omega_{n,m}^\circ &= 2\sqrt{1 + \vec{e}_{\alpha,n} \vec{e}_{\alpha,m}} \hat{n}_n S_{n,m} = \\ &= 2\sqrt{1 + \vec{e}_{\alpha,n} \vec{e}_{\alpha,m}} S_{n,m} \hat{n}_m. \end{aligned}$$

and therefore, denoting the mean value of the normalization factor by B , one may write

$$2 \left(-\frac{1}{16}\right)^N \text{tr} P \prod_c \Omega^\circ = 2\phi B^{2N}$$

Here we have assumed that when calculating the integral over \vec{e}_α one may write instead of the product of normalization factors $\prod 2\sqrt{1 + \vec{e}_\alpha \vec{e}_\alpha}$ the mean value of the factor B with the power L (the length of the contour). Since, by our construction,

$$L = 2N = \frac{A}{2\epsilon^2}$$

one has, at least in the leading divergency approximation,

$$\left\langle \left(-\frac{1}{16}\right)^N \prod_c 2\sqrt{1 + \vec{e}_{\alpha,i} \vec{e}_{\alpha,i+1}} \right\rangle \sim e^{\epsilon n B \frac{A}{2\epsilon^2}}$$

The diagrams of Fig. 11b give the expression

$$\prod_{\text{plaquettes}} [2 - \frac{1}{2}(1 + \vec{e}_n \vec{e}_m)] \sim \mathcal{D}^N$$

The quantity \mathcal{D} is positive.

The final expression for the free energy is

$$F = F_0 + F_1 \quad (45)$$

where

$$F_0 = \int \mathcal{D}_x e^{-m_0^2 \int d^2 \xi \sqrt{g}} \quad (46)$$

$$m_0^2 = m^2 - \frac{1}{4\epsilon^2} \ln \mathcal{D}$$

is the bosonic part, and

$$-F_1 = \int \mathcal{D}_x e^{-m_1^2 \int d^2 \xi \sqrt{g}} 2\phi \quad (47)$$

$$m_1^2 = m^2 - \frac{1}{2\epsilon^2} \ln B$$

is the fermionic part coinciding with the free energy of 3d IM in continuum limit. Note that our method of calculation (e.g. the way we discretized the action) was chosen only for the reasons of convenience. In the other methods of calculation the surface must be covered by the contours in all possible ways. Then an additional factor in the functional integral will arise, equal to the number of possible coverings. We believe that this factor will be proportional to e^{-cA} and will lead to some additional renormalization of the string tension.

Conclusions

We have pointed out the possibility of the fact that the free energy of the superstring is equal to the sum of the free energies of bosonic and fermionic strings. The latter coincides with the free energy of 3d IM near the critical point. To investigate the critical behaviour of 3d IM one has

to quantize the superstring and to extract the fermionic part of its spectrum. It is reasonable to expect that the fermionic component of the superstring coincides with Neveu-Schwarz-Polyakov fermionic string.

The investigation of 3d IM with the use of fermionic variables was also made in Refs. [14, 15].

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APPENDIX

Here we will show by explicit calculation that if the contour does not enclose the singular point - the end of the line of selfintersection, - then $\chi(c) = 1$. It is sufficient to consider the contour shown in Fig.12. Here we use the vectors rotated by $\pi/4$ with respect to the vectors described in the text. The sign factor does not feel this change owing to the property (14)

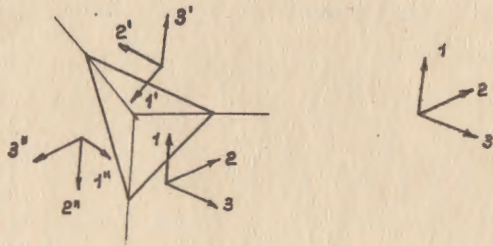


Fig. 12

The calculation goes as follows:

$$\begin{aligned} \chi(c) &= -\frac{1}{2} \text{tr} \Pi S e e' S e' e'' S e'' e = \\ &= -\frac{1}{2} \text{tr} \{ (1 + \hat{e}_\alpha \hat{e}'_\alpha) (1 + \hat{e}'_\alpha \hat{e}''_\alpha) (1 + \hat{e}''_\alpha \hat{e}_\alpha) \} \\ &= \frac{1}{2^3 \sqrt{(1 + \bar{e}_\alpha \bar{e}'_\alpha) (1 + \bar{e}'_\alpha \bar{e}''_\alpha) (1 + \bar{e}''_\alpha \bar{e}_\alpha)}} = \\ &= -\frac{1}{2} \text{tr} (1 - \hat{e}_1 \hat{e}_2 - \hat{e}_2 \hat{e}_3 + \hat{e}_3 \hat{e}_1)^3 \cdot \frac{1}{2^3} = 1. \end{aligned}$$

We have used the relation $\hat{e}_\alpha \hat{e}_\beta = i \epsilon_{\alpha\beta\gamma} \hat{e}_\gamma$, ($\alpha \neq \beta$)

Now consider the contour shown in Fig.13.

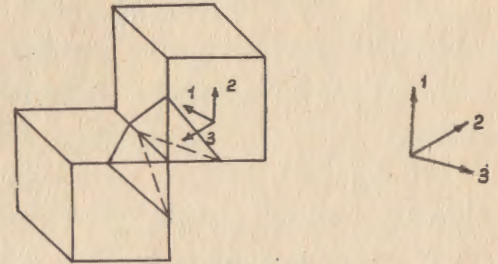


Fig. 13

The surface here is such that the selfintersection may be removed by small deformation. To compute the sign factor we start from the point 0 and write the product of the rotation matrices moving in the direction shown by the arrow:

$$\begin{aligned} \chi(c) &= -\frac{1}{2} \text{tr} \{ (1 + \hat{e}_2 \hat{e}_1 - \hat{e}_1 \hat{e}_3 - \hat{e}_3 \hat{e}_2) \\ & (1 - \hat{e}_1 \hat{e}_3 + \hat{e}_3 \hat{e}_2 - \hat{e}_2 \hat{e}_1) (1 + \hat{e}_3 \hat{e}_2 + \hat{e}_2 \hat{e}_1 + \hat{e}_1 \hat{e}_3)^2 \\ & (1 + \hat{e}_1 \hat{e}_3 - \hat{e}_3 \hat{e}_2 - \hat{e}_2 \hat{e}_1) (1 - \hat{e}_3 \hat{e}_2 + \hat{e}_2 \hat{e}_1 - \hat{e}_1 \hat{e}_3) \} \frac{1}{2^6} = 1 \end{aligned}$$

For the contour shown in Fig.14

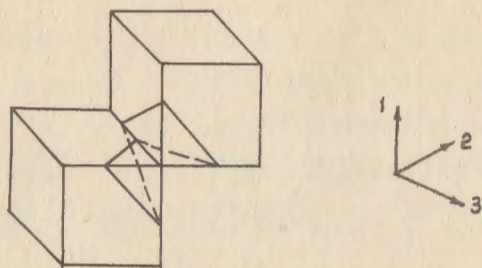


Fig. 14

$$\chi(c) = -\frac{1}{2} \text{tr} \left\{ (1 + \hat{e}_2 \hat{e}_1 - \hat{e}_1 \hat{e}_3 - \hat{e}_3 \hat{e}_2)^2 (1 - \hat{e}_1 \hat{e}_3 + \hat{e}_3 \hat{e}_1 + \hat{e}_3^2) \right.$$

$$\left. (1 + \hat{e}_3 \hat{e}_1 + \hat{e}_1 \hat{e}_2 + \hat{e}_2 \hat{e}_3)^2 (1 + \hat{e}_2 \hat{e}_3 - \hat{e}_3 \hat{e}_2 + \hat{e}_1^2) \right\} \frac{1}{2^8} = -1$$

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