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ЦЕНТРАЛЬНЫЙ НАУЧНО-ИССЛЕДОВАТЕЛЬСКИЙ ИНСТИТУТ ИНФОРМАЦИИ И ТЕХНИКО-ЭКОНОМИЧЕСКИХ ИССЛЕДОВАНИЙ ПО АТОМНОЙ НАУКЕ И ТЕХНИКЕ

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ON A DIAGRAM TECHNIQUE IN A FIELD THEORY AT FINITE TEMPERATURE

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Introduction

The aim of this work is the development of the diagram technique for the calculation of the partition function, effective potential, Green functions in relativistic field theory at finite temperature. The consideration of the theory at finite (i.e. nonzero) temperature means that one has to replace vacuum averaging by the averaging over all excited states with the Hibbs weight:

$$\langle 0|\ldots|0\rangle \to \frac{1}{Z}\sum_{n}\langle n|\ldots|n\rangle e^{-\frac{E_n}{T}} = \frac{1}{Z}Sp\left(e^{-\frac{H}{T}}\ldots\right)$$

where $Z = Sp\, e^{-H/T}$, Z is the partition function.

The methods for calculating such statistical averages were studied intensively in the 50-ies, when the Feynman diagram technique in quantum electrodynamics had just appeared. The generalization of the latter for the $T \neq 0$ case was carried out and intensively used in the works of Matsubara [1] , Abrikosov, Gorkov and Dzyaloshinsky [2] , Fradkin [3] and the others, wherein it was usually used in the relativistic situation.

Relativistic, in particular gauge theories at finite temperature were studied in the 70-ies in the works of Kirzshniz and Linde [4] , Weinberg [5] , Dolan and Jackiw [6] , Bernard [7] , who actually applied the statistical physics ideas to relativistic field theory. Usually in these works the Matsubara technique was used, but also the other ones were developed, which do not use imaginary time and energy, e.g. those of Mills [8] , Keldysh [9] , Dolan and Jackiw [6] , Takahashi and Umezawa [10] .

The diagram technique suggested in the present work resembles that of Dolan and Jackiw, but differs from it. Namely we show that the diagrams of Dolan and Jackiw contain some parts which must not be taken into account (a more detailed comparison see in Sec.1).

In Sec.1 we explain our technique in detail. In Sec.2 we, for completeness, give an account of the Matsubara technique and also explain our method for proving the equivalence of two considered formalisms. The method is straightforward: we explicitly compute, for an arbitrary diagram, the Matsubara sum over energies (Sec.3) and integrals over K_0 (Sec.4) in our formalism. The results will be the same.

In the Conclusion we discuss some applications of our formalism.

1. The Diagram Technique in the Real Time

We shall describe the diagram technique suggested by us for the calculation of the free energy (and effective potential) firstly in the simplest case of the theory of the self-interacting scalar field. It is well known that the value of the free energy at zero temperature is given by the sum of all connected vacuum (i.e. without external legs) diagrams. The vertices in those diagrams are given by the interaction lagrangian, and the propagators are Feynman causal ones:

$$(A)=\frac{i}{K^2-m^2+i\varepsilon}$$

Our prescription is as follows: for the computation of the free energy at nonzero temperature $T\ (=1/\beta)$ one has to consider the same sum of $T=0$ diagrams, but instead of each diagram to take a sum of diagrams which differs from the initial one in that the propagators in the new diagrams are equal either to (A) or to

$$(B)=2\pi\delta(K^2-m^2)\,n(E_K)\,,\quad n(E_K)=\frac{1}{e^{\beta E_K}-1}\,,\quad E_K=\sqrt{\vec{K}^2+m^2}$$

In the course of generating the new diagrams the following important rule must hold: each of the new diagrams has to remain connected after the cutting of all its (B) type propagators. Diagrams that become disconnected have not to be included. Let's give an example for the φ^3 theory. The diagram of Fig.1

Fig.1

at $T \neq 0$ has to be replaced by the sum of diagrams in Fig.2

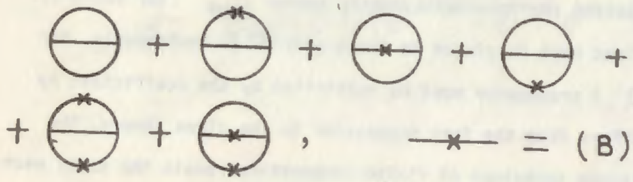

Fig.2

The possible diagram of Fig.3 is discarded since it becomes disconnected after cutting all its (B) type propagators. (Although in this particular

Fig.3

case the diagram in Fig.3 is zero, it is not so in general - for example the similar diagram in the φ^4 theory (Fig.4) is nonzero).

Fig.4

For theories with fermions rules remain the same, but the corresponding (A) and (B) propagators are

$$(A') = i\,\frac{\hat{K}+m}{K^2-m^2+i\varepsilon}$$

$$(B') = 2\pi\,\delta(K^2-m^2)\,f(E_K), \qquad f(z) = 1/(e^{z\beta}+1)$$

For the arbitrary relativistic theory (gauge theories we take in Feynman gauge) free propagators of bosonic field may differ from (A) only by the coefficient reflecting the internal symmetry and/or $g_{\mu\nu}$ (for vectors). The same coefficient must be placed in front of (B). Analogously, for fermions the (B') propagator must be multiplied by the coefficient by which (A') differs from the free propagator in the given theory. The rules for the diagram technique at finite temperature remain the same: each

vacuum diagram generates a set of diagrams with all possible (B) type propagators insertions, with the restriction that diagrams have to remain connected after cutting all their (B) type propagators. But for this last limitation, the whole story would evidently have reduced to calculating the same vacuum diagrams with propagators equal to $(A) + (B)$. This is the prescription of Dolan and Jackiw [6] and it evidently differs from ours

Let's discuss a technique for the calculation of causal Green functions. At the moment we have no complete results, but remembering that variations of the free energy are connected with the Green functions, we formulate a hypothesis which is proved by now only in particular cases: the calculation of the causal (time ordered) Green functions must be carried out by the same rules as above, but the restriction on the inclusion of the (B) type propagators is the following. The diagram has to remain connected after glueing all external lines into one new vertex and cutting all (B) type propagators.

The author supposes that the drawback in the considerations of [6] is in that their boundary conditions on Green functions are set in the complex time (t) points, while causal Green functions are not analytical functions of t . For example, the two-point function in the (1+0) dimensional theory of free massive scalar field is

$$\frac{1}{2m}\,\frac{1}{e^{-\beta m}-1}\,(-1)\left(e^{-im|t|} + e^{-m\beta}\,e^{im|t|}\right)$$

Note also, that infinities which arose in [6] in diagrams like those in Fig.5a disappear in our formalism: divergent is the diagram of Fig.5b, which according to the rules of Sec.1 must be discarded.

a) b)

Fig.5

2. The Matsubara Imaginary-Time Technique

The Matsubara technique for finite-temperature calculations was discussed in the text-books (see, e.g. [11.12]). We shall now describe it briefly, in momentum-space representation, for the free energy calculation.

Again, as in Sec.1, one has to consider a sum of all vacuum diagrams, but there is no need to construct new diagrams (with (B) type propagators), Instead, in usual propagators

$$\frac{i}{k^2-m^2} \qquad \text{for bosons}$$

$$i\,\frac{\hat{k}+m}{k^2-m^2} \qquad \text{for fermions}$$

integration has to be replaced by the summation over the following pure imaginary values of

$$k_0 = 2n\pi i/\beta \qquad \text{for bosons}$$

$$k_0=(2n+1)\pi i/\beta \qquad \text{for fermions}, \quad n=0,\pm 1,\pm 2,\dots$$

In each vertex the three-momentum and the discrete imaginary energy are compared. The method to prove that the sum of diagrams for the free energy considered in Sec.1 is equal to the sum of the Matsubara diagrams for the same quantity is given below. Actually, we shall prove that each imaginary-time vacuum diagram is equal to the sum of diagrams which it generates in real-time formalism of Sec.1. Note also, that in both cases we discard common multipliers of all vacuum diagrams - the space-time volume (they differ in that the infinite time interval in one case is replaced by the finite imaginary one ($i\beta$) in the other), and in fact consider the density of the free energy. We shall explicitly compute, in arbitrary vacuum diagram, all sums over discrete imaginary energies in imaginary-time formalism and all energy integrals in the corresponding sum of diagrams of real-time formalism. The result will be the same.

So, we shall not at all consider the integrals over the space components of momentum. In all propagators we make the relabelling of the following type: $\vec{k}^2+m^2 \equiv m_1^2$, so the denominator of each propagator becomes like $k_0^2-m_1^2$. We shall also omit the subscript 0 in k_0 . Strictly speaking, all the following calculations are completely correct, if masses (m_1) in all propagators are kept slightly different and take their initial values only in the finite answer (to avoid singularities like $\delta(k^2-m^2)/(k^2-m^2)$, which actually cancel with the other similar ones).

Energy integrals are convergent even in the vacuum diagrams - it is so in most of the renormalizable theories, - so we do not introduce a regularization.

3. The Sums

In this section we shall calculate explicitly the sums over energies in the arbitrary vacuum diagram, firstly for the pure bosonic theory. The modifications for accounting of theories with fermions are trivial and are considered at the end of this section.

So let us consider the vacuum diagram. Bearing in mind the conservation of the four-momentum in vertices, the momentum distribution in propagators is the following: some propagators have momenta $K_1, K_2, \dots K_N$, N is the number of loops in the diagram, other propagators carry the momentum which is defined by the conservation laws in vertices. It is evident, that the cutting of n ones from the propagators with $K_1, \dots, K_N$ momenta leads to the diagram with $N-n$ independent loops. Other propagators carry momenta which are some linear combinations of $K_1, \dots, K_N$. Later on, we shall need the following statement concerning these linear combinations:

1) each of these combinations contains K_i with the coefficient 0, 1, or -1;

2) if we equate one of these combinations to zero, then express one K_i through the others, and substitute the result into all remaining propagators, then these new combinations of K_i will satisfy the same conditions 1) and 2).

We omit the non-difficult proof of this statement.

Summations over K_i we shall carry out in the following succession: first over K_1, then over K_2, and so on. We use the well-known formula (see, e.g. [12]):

$$\frac{i}{\beta}\sum_{z=2n\pi i/\beta} \varphi(z) = \frac{1}{i}\sum_{z_K} \operatorname{res}_{z_K}\left(\beta(z)\varphi(z)\right) \tag{1}$$

z_K are poles of $\varphi(z)$. (We assume that the sum is convergent.)

In our case the function φ is given essentially by

$$\varphi(K_1, \dots K_N) = \prod_a \frac{1}{(P_a^2 - M_a^2)} \tag{2}$$

Index a runs over all propagators. P_a is the momentum of the a-th propagator, it is a linear combination of K_i. "Essentially" means that we discard polynomials of momenta possibly existing in vertices. They are inessential when using (1) and subsequent calculations.

We are now ready for the calculation of

$$X = \left(-\frac{1}{\beta}\right)\sum_{K_N}\left(-\frac{1}{\beta}\right)\sum_{K_{N-1}} \dots \left(-\frac{1}{\beta}\right)\sum_{K_1}\varphi(K_1 \dots K_N). \tag{3}$$

Summing over K_1 by means of (1) we get

$$X = \left(-\frac{1}{\beta}\right)^{N-1}\sum_{K_2 \dots K_N}\ \sum_{\substack{a_1 \\ x_{a_1} = \pm M_{a_1}}} \frac{\alpha_1}{2x_{a_1}}\,\beta(\alpha_1 x_1)\,. \tag{4}$$

$$\prod_{a \neq a_1} \frac{1}{(P_a')^2 - M_a^2}$$

a_1 runs over all propagators for which P_{a_1} depends on K_1:

$$P_{a_1} = \alpha_1 K_1 + \dots, \alpha_1 = \pm 1$$

P_a' is P_a in which K_1 is expressed by the $K_2 \dots K_N$ from $P_{a_1} = x_{a_1}$

In (4), it is taken into consideration that $K_2 \dots K_N$ still have their discrete imaginary values, so the value of $\beta(K_1)$ at the point $P_{a_1} = x_{a_1}$, which is equal to $\beta(\alpha_1 x_1 + \gamma_i K_i)$, $\gamma_i = 0, \pm 1, i > 1$

is equal to $\delta(\alpha_1 x_1)$. Each term in the sum over α_1 in (4) looks like (3) with $N \to N-1$. Using (1) again, for the summation over K_2 we get

$$X = \left(-\frac{1}{\beta}\right)^{N-2} \sum_{K_3 \dots K_N} \sum_{\substack{\alpha_1 \\ x_{\alpha_1} = \pm M_{\alpha_1}}} \frac{\alpha_1}{2x_{\alpha_1}} \delta(\alpha_1 K_1) \cdot$$

$$\cdot \sum_{\substack{\alpha_2 \\ x_{\alpha_2} = \pm M_{\alpha_2}}} \frac{\alpha_2}{2x_{\alpha_2}} \delta(K_2 = \dots) \prod_{\alpha \neq \alpha_1, \alpha_2} \frac{1}{(p_\alpha'')^2 - M_\alpha^2} \tag{5}$$

Notations are similar to those in (4): α_2 runs over all propagators for which p_{α_2}' depends on K_2: $p_{\alpha_2}' = \alpha_2 K_2 + \dots$, $\alpha_2 = \pm 1$; p_α'' is the p_α in which K_1 and K_2 are expressed by K_i, $i > 2$, from the equations $p_{\alpha_1} = x_{\alpha_1}$, $p_{\alpha_2} = x_{\alpha_2}$.

The argument of the δ -function in $\delta(K_2 = \dots)$ is the value of K_2 found from the equation $p_{\alpha_2}' = x_{\alpha_2}$. For example, if $p_{\alpha_1} = K_1 + K_2$, $p_{\alpha_2} = K_1$, then $p_{\alpha_2}' = x_{\alpha_1} - K_2$ and $\delta(K_2 = \dots) = \delta(x_{\alpha_1} - x_{\alpha_2})$.

In the same way we carry out summation over $K_3 \dots K_N$. The final form of the formula is

$$X = \sum_{a, x_a} \frac{\alpha}{2x_a} \delta(K_1 = \dots) \sum_{b, x_b} \frac{\beta}{2x_b} \delta(K_2 = \dots) \cdot \dots \cdot$$

$$\cdot \sum_{c, x_c} \frac{\gamma}{2x_c} \delta(K_N = \dots) \cdot \prod_{a' \neq a, b, \dots, c} \frac{1}{(p_{a'}^{(N)})^2 - M_{a'}^2} \tag{6}$$

Here the number of $a, b, \dots, c$ is N . $p_{a'}^{(N)}$ is the $p_{a'}$ in which instead of $K_1, \dots K_N$ their values from the equations $p_a = x_a$, $p_b = x_b$ . . . , $p_c = x_c$ are substituted. Index a runs over all propagators in which p_a depends on K_1 ; b runs over all propagators in which p_b' depends on K_2 ; ...,c runs over all propagators for which $p_c^{(N-1)}$ depends on K_N ; $\alpha = \pm 1$ is the sign of K_1 in p_a , $\beta = \pm 1$ is the sign of K_2 in p_b', . . . , $\gamma = \pm 1$ is the sign of K_N in $p_c^{(N)}$. Arguments of δ -functions are the following: K_1 is from the equation $p_a = x_a$ (and, as we have noted previously, in the argument of $\delta(K_1 = \dots)$ all K_i , $i > 1$ may and have to be omitted); K_2 is from the equation $p_b' = x_b$ (and K_i , $i > 2$ have to be omitted) and so on. Evidently, the arguments of δ -functions are linear combinations of $x_a, x_b, \dots x_c$. It may be easily proved, that these linear combinations satisfy the same conditions 1) and 2) as p_i do. It is also evident, that after cutting al $a, b, \dots, c$ propagators we shall get a connected tree (i.e. without loops) diagram.

Let's choose a set of propagators $\bar{a}, \bar{b}, \dots, \bar{c}$, which satisfy this last condition, and choose also a set $x_{\bar{a}}, x_{\bar{b}}, \dots, x_{\bar{c}}$, ($x_{\bar{a}} = \pm M_{\bar{a}}$, $x_{\bar{b}} = \pm M_{\bar{b}}$, $x_{\bar{c}} = \pm M_{\bar{c}}$). Let's consider in the sum (6) all terms for which sets $(a, b, \dots, c)$ and $(x_a, x_b, \dots, x_c)$ coincide, maybe after transpositions with the sets $(\bar{a}, \bar{b}, \dots, \bar{c})$ and $(x_{\bar{a}}, x_{\bar{b}}, \dots, x_{\bar{c}})$ respectively.

Let's denote the sum of all such terms by the $X(\bar{a}, \dots, x_{\bar{a}}, \dots)$. Clearly

$$X = \sum_{\substack{(\bar{a}, \bar{b}, \dots, \bar{c}) \\ (x_{\bar{a}}, x_{\bar{b}}, \dots, x_{\bar{c}})}} X(\bar{a}, \bar{b}, \dots, \bar{c}; x_{\bar{a}}, x_{\bar{b}}, \dots, x_{\bar{c}}) \tag{7}$$

The sum here is over all choices of a set $(\bar{a}, \bar{b}, \dots \bar{c})$ from all propagators, and over all choices of signs in equations $x_{\bar{a}} = \pm M_{\bar{a}}$, ... , $x_{\bar{c}} = \pm M_{\bar{c}}$

Consider $X(\bar{a}, \bar{b}, \dots, \bar{c}; x_{\bar{a}}, x_{\bar{b}}, \dots x_{\bar{c}}) \equiv X(\bar{a}, \dots; x_{\bar{a}} \dots)$

In different terms entering into $X(\bar{a},\dots;X_{\bar{a}}\dots)$, $K_1,K_2,\dots,K_N$ always have to be found from the same system of equations; denote it Δ :

$$
\begin{aligned}
P_{\bar{a}} &= X_{\bar{a}}\\
P_{\bar{b}} &= X_{\bar{b}}\\
&\dots\\
P_{\bar{c}} &= X_{\bar{c}}
\end{aligned}
\tag{8}
$$

Different terms differ only in the succession of using Eqs.(8) for finding $K_1, K_2, \dots, K_N$

It is easy to understand that $X(\bar{a},\dots;X_{\bar{a}}\dots)$ may be represented in the following form:

$$
X(\bar{a},\dots;X_{\bar{a}}\dots)=\frac{1}{2X_{\bar{a}}}\,\frac{1}{2X_{\bar{b}}}\cdots\frac{1}{2X_{\bar{c}}}\prod_{\alpha\neq\bar{a},\dots\bar{c}}\frac{1}{(P_\alpha^{(N)})^2-M_\alpha^2}\cdot F^{\Delta}(x)
\tag{9}
$$

$$
F^{\Delta}(x)=\sum_{(\bar{a},\bar{b},\dots,\bar{c})}\alpha\delta(K_1=\dots)\beta\delta(K_2=\dots)\cdot\dots\cdot\gamma\delta(K_N=\dots)
\tag{10}
$$

$\sum_{(\bar{a},\bar{b},\dots\bar{c})}$ means the sum over all permutations of indices $\bar{a},\bar{b},\dots,\bar{c}$

For example, if permutation is $(\bar{a},\bar{b},\dots,\bar{c})\to(\bar{b},\bar{c},\dots\bar{a})$, then K_1 in $\delta(K_1=\dots)$ have to be defined from the equation $P_{\bar{b}}=X_{\bar{b}}$, then K_2 - from the $P'_{\bar{c}}=X_{\bar{c}}$, and K_N -- from the $P_{\bar{a}}^{(N-1)}=X_{\bar{a}}$. If for some permutation one of these equations doesn't contain the corresponding K_i , we suppose that term zero. The key point of our calculations is the

following

Statement :

$$
F^{\Delta}(x)=\prod_{i=\bar{a},\bar{b},\dots,\bar{c}}\lambda_i\delta(\lambda_i X_i)
\tag{11}
$$

Here $\lambda_i=\pm 1$ being defined as follows.

Let's consider the solution of system (8):

$$
\begin{aligned}
K_1 &= \dots\\
K_2 &= \dots\\
&\dots\\
K_N &= \dots
\end{aligned}
\tag{12}
$$

We have mentioned earlier that in r.h.s. of (12) X_i enter with coefficients 0 or ±1. The sign λ_i is the one with which X_i initially appears in r.h.s. of Eqs.(12) if considering them from the bottom to the top.

We now give an example with $N=2$ Let we have equations

$$
P_{\bar{a}}=K_1=X_1 \tag{13}
$$

$$
P_{\bar{b}}=K_1+K_2=X_2 \tag{13'}
$$

The sum over permutations in F

$$
F=\sum_{(\bar{a},\bar{b})}\alpha\delta(K_1=\dots)\beta\delta(K_2=\dots)
\tag{14}
$$

consists of two terms.

The first one is the identical permutation $(\bar{a}\bar{b})\to(\bar{a}\bar{b})$, hence we find K_1 and α from (13): $K_1=X_1$, $\alpha=1$. then substitute K_1 in (13') and find K_2 and β : $K_2=X_2-X_1$, $\beta=1$. Accordingly, the first term in (1) is $\delta(X_1)\delta(X_2-X_1)$. Analogously, for the transposition $(\bar{a}\bar{b})\to(\bar{b}\bar{a})$ the corresponding term in (14) is

$-\theta(x_2)\theta(x_2-x_1)$. It can directly be checked that

$$F=\theta(x_1)\theta(x_2-x_1)-\theta(x_2)\theta(x_2-x_1)=-\theta(-x_1)\theta(x_2) \tag{15}$$

On the other hand, system (13), (13') has the solution

$$K_1 = x_1 \tag{16}$$

$$K_2 = x_2 - x_1 \tag{16'}$$

According to the rule for finding λ_i we get (already from (16')), that $\lambda_1=-1$ and $\lambda_2=1$, and according to the statement (11)

$$F=(-1)\,\theta(-x_1)\,\theta(x_2)$$

which is in agreement with (15).

The general proof will be carried out by the induction over N . At $N=1$, the statement is evident. Let's suppose it true at $N-1$ and consider the system of N equations (8).

Let I be a subset of a set $N=\{\bar{a},\bar{b},\dots,\bar{c}\}$, such that p_i depend on K_i iff $i\in I$: $p_i=\alpha_i K_1+\dots$, $\alpha_i=\pm 1$.
The sum over permutations in (10) we shall rewrite now as a double sum: a sum over all possible choices of a first element, and at each of such choices a sum over all permutations of the remaining elements. In other words, we choose in all possible ways from (8) the equation for finding K_1 and note that at each of such choices the system of the remaining $N-1$ equations has to be worked up by the same procedure as in (10). The result of the latter is known by the induction. So, we shall write now a recurrence relation for F . Note only the following. If we define K_1 from the i-th equation and substitute it in all the others, then the remaining system

Δ_i' of $N-1$ equations will have the following r.h.sides (after transporting x_i from l.h.s. to r.h.s.).

$$\tilde{x}_k=\begin{cases} x_k-\alpha_k\alpha_i x_i\,, & k\in I,\ k\neq i\\ x_k\,, & k\in N-I\end{cases} \tag{17}$$

Here is the recurrence relation:

$$F^{\Delta}(x)=\sum_{i\in I}\alpha_i\,\theta(\alpha_i x_i)\,F^{\Delta_i'}(\tilde{x}) \tag{18}$$

$F^{\Delta_i'}$ is given (by induction) by the statement (11):

$$F^{\Delta_i'}(\tilde{x})=\prod_{k\in N,k\neq i}\lambda_k\theta(\lambda_k\tilde{x}_k)=\prod_{j\in I,j\neq i}\lambda_j\theta(\lambda_j(x_j-\alpha_j\alpha_i x_i))\cdot \tag{19}$$

$$\cdot\prod_{k\in N-I}\lambda_k\theta(\lambda_k x_k)$$

Signs λ_i here should have been defined by the solution of the system Δ_i' , but it is evident that they may equally well be defined from Δ .

Combining (18) and (19) we get

$$F^{\Delta}(x)=\sum_{i\in I}\alpha_i\,\theta(\alpha_i x_i)\prod_{j\in I,j\neq i}\lambda_j\theta(\lambda_j(x_j-\alpha_j\alpha_i x_i))\cdot \tag{20}$$

$$\cdot\prod_{k\in N-I}\lambda_k\theta(\lambda_k x_k)$$

We use now the identity (A1) (see Appendix) at $æ=\alpha_i\lambda_i$, $y_i=\alpha_i x_i$, and also the fact that all $x_i\alpha_i$ cannot be equal to (-1) simultaneously, and get the desired answer

$$F^{\Delta}(x) = \prod_{\kappa \in N} \lambda_\kappa \theta(\lambda_\kappa x_\kappa)$$

Substitute it in (9) and use (at real x)

$$\frac{1}{2x}\delta(x) = \frac{1}{2|x|}\cdot(\delta(|x|) + \theta(-x))$$

to get the answer for $X(\bar{a}, \dots ; x_{\bar{a}}, \dots)$:

$$X(\bar{a},\dots;x_{\bar{a}},\dots) = \prod_{a \neq \bar{a},\dots,\bar{c}} \frac{1}{(P_a^{(N)})^2 - M_a^2} \cdot \tag{21}$$

$$\cdot \prod_{a=\bar{a},\dots,\bar{c}} \frac{1}{2M_a}\left(\delta(M_a) + \theta(-\lambda_a x_a)\right)$$

4. Calculation of the Corresponding Sum of Diagrams in the Real-Time Formalism

Let's consider now a sum of diagrams which are generated by the diagram of preceding section in the real-time formalism according to the rules of Sec.1. In each of these real-time formalism diagrams (which are topologically identical to the diagram considered in Sec.3 and differ only in propagators) the momentum flowing through the a -th propagator is p_a as in Sec.3. We also integrate over momenta in the same order: firstly over K_1 , then over K_2 and so on. Integration over K_i is carried out either by the δ -function from (B) type propagator or we continue integration contour (real axis) by the upper semicircle and the integration is reduced to the calculation of the residues of the function under integration. In both cases the system defining K_i is the same, because δ -function $\delta(K_i^2 - m^2)$

gives the same equations for K_i : $K_i = \pm M$ as those from the poles of the propagator $1/(K_i^2 - M^2)$.

From the considered set of diagrams we now pick out the one which has its (B) type propagators with indices from the subset I of the set $N = \{\bar{a}, \bar{b}, \dots, \bar{c}\}$ (notations are taken from the preceding section). Many different terms arise in the process of integration. We consider those in which the system defining $K_1, \dots, K_N$ coincides with (8). Evidently, all those terms have a multiplier as in (21):

$$\prod_{a \neq \bar{a},\dots,\bar{c}} \frac{1}{(P_a^{(N)})^2 - M_a^2} \cdot \prod_{a=\bar{a},\dots,\bar{c}} \frac{1}{2M_a} \tag{22}$$

Besides, there is a multiplier

$$\prod_{i \in I} \delta(M_i) \tag{23}$$

The remaining multiplier is, as we shall prove now,

$$Y(\bar{a},\dots;x_{\bar{a}}\dots) = \prod_{n \in N-I} \theta(-\lambda_n x_n) \tag{24}$$

It coincides with the coefficient at the same ((23)) product of δ -functions in (21).

Analogously, all the other products of δ -functions in (21) exactly arise from the other diagrams of real-time formalism - namely from diagrams with the other distributions of (B) type propagators over $\bar{a}, \bar{b}, \dots, \bar{c}$ propagators.

Thus, we see that the sum of real-time diagrams with all posible distributions of (B) type propagators over propagators $\bar{a}, \bar{b}, \dots, \bar{c}$ is exactly the sum of $X(\bar{a},\dots; x_{\bar{a}}\dots)$ over all sets $x_{\bar{a}}, \dots x_{\bar{c}}$. Due to relation (7) we see that the Matsubara diagram of Sec.3 is equal to the sum of real-time diagrams, which it generates according to the rules of Sec.1.

It remains to prove the statement concerning (24).

We consider a diagram having (B) type propagators with indices from I and pick out terms in which integration yields for $K_1, \ldots, K_N$ the system (8). Evidently they all have the same multipliers (22) and (23). Besides, the sum of all such terms must be equal to (24). This we shall prove by induction over N .

Suppose it true for $N-1$ and consider a system of N equations (8). Integration over K_1 may be carried out either by δ-function (if one of P_i , $i \in I$, depends on K_1) or, as described earlier, is reduced to the calculation of residues in upper semiplane. First, consider the second case. Taking into account that imaginary parts of masses are negative ($M - i\varepsilon$) and using the induction, it is easy to understand that

$$Y(\bar{a}, \ldots ; x_{\bar{a}}, \ldots) = \sum_{\ell} \Big\{ \theta(-\alpha_\ell x_\ell) \cdot \alpha_\ell \cdot \operatorname{sign} x_\ell \cdot \prod_{\substack{n \in N-I \\ n \neq \ell}} \theta(-\lambda_n (x_n - \alpha_n \alpha_\ell x_\ell)) \Big\} \tag{25}$$

ℓ runs over all indices from N such that P_ℓ depends on K_1 :
$P_\ell = \alpha_\ell K_1 + \ldots, \alpha_\ell = \pm 1$

Different sign factors in (25) have simple explanation: $\theta(-\alpha_\ell x_\ell)$ means that poles have to lie in upper semiplane; α_ℓ is the sign of the residue, sign x_ℓ arises due to the fact that in (22) $1/2|x_\ell|$ is carried away, while in the residue (of $1/(P_\ell^2 - x_\ell^2)$) $1/2x_\ell$ arises. The last factor which is equal to $Y^{\Delta_\ell'}(\ldots;\ldots)$ is written through θ-function

by induction. (25) yields

$$Y(\bar{a}, \ldots ; x_{\bar{a}}, \ldots) = -\sum_{\ell} \theta(-\alpha_\ell x_\ell) \cdot \prod_{\substack{n \in N-I \\ n \neq \ell}} \theta(-\lambda_n (x_n - \alpha_n \alpha_\ell x_\ell)) =$$

$$= \prod_{n \in N-I} \theta(-\lambda_n x_n)$$

The last equality is the $\beta \to \infty$, $\varkappa_i = \lambda_i \alpha_i$, $y_i = \alpha_i x_i$ limit of (A1) and coincides with the promised one (24).

In the other case, when δ-function exists with argument P_ℓ , dependent on K_1 , we use it for the integration over K_1 and get at once by induction.

$$Y(\bar{a}, \ldots ; x_{\bar{a}} \ldots) = \prod_{n \in N-I} \theta(-\lambda_n x_n)$$

It is evident, since in the δ-function mass x_ℓ has no imaginary part. after expressing K_1 by $K_2 \ldots K_N$, substituting in P_i , $i \neq \ell$ and carrying from the left to the right (in (8)), all poles remain in their semiplane. This completes the proof of (24) and, in general, of the whole statement about equivalence of the Matsubara and real-time formalism of Sec.1.

We shall now discuss briefly the proof of this statement for theories with fermions.

The proof reduces to the one for pure bosonic theories, if one notes that fermionic distribution function $f(z)$ in (B') may be received from bosonic one $b(z)$ by translation of z on $\pi i/\beta$:

$$f(z) = -b\left(z + \frac{\pi i}{\beta}\right)$$

(The minus sign is also in agreement with ($\mathcal{B}$) and ($\mathcal{B}'$)). Therefore, at a suitable moment (when considering (8)) we may consider all K_i bosonic, and shift X_i for fermionic propagators by $\pi i/\beta$. In all other respects, the proof is completely analogous and we don't write it here.

Conclusion

Thus, the formalism of Sec.1 allows one to calculate the free energy of arbitrary relativistic theory at finite temperature. It is easy to understand, that one may use it also for the calculation of the effective potential for scalar field. This follows from the fact, that calculation of the latter in the Matsubara formalism reduces to the one of the same vacuum diagrams with propagators with new masses (dependent on the values of scalar fields). The proof of Secs.3, 4 shows that we may calculate the same quantity by the rules of Sec.1, putting new masses in ($\mathcal{A}$) and ($\mathcal{B}$) propagators.

Let's discuss in what cases this formalism has some advantages compared with the Matsubara one. Firstly we note that this technique is convenient for the low-temperature expansion. Indeed, at low temperatures ($\beta \to \infty$) $\text{б}(E) \sim \exp(-m/T)$ and ($\mathcal{B}$) type propagators are small, so in some sense, the number of ($\mathcal{B}$) type propagators in the diagrams means an order of zero-temperature expansion. It may be proved that combining together the diagrams with one crossed (($\mathcal{B}$) type) propagator we get the amputated two-point zero-temperature causal Green function; the sum of diagrams with two crossed (($\mathcal{B}$) type) propagators gives an amputated four-point zero-temperature casual function, and so on. Consequently, free energy has the following representation

$$F(T) = F(T=0) + \operatorname{Sp} \ln \mathcal{D}_T/\mathcal{D}_0 + \int dp\, 2\pi\,\delta(p^2-m^2)\, G_2(p)\, \text{б}(E_p)$$

$$+ \frac{1}{2!\,2^2} \int dp\, dq\, (2\pi)^2\, \text{б}(E_p)\, \text{б}(E_q)\, \delta(p^2-m^2)\, \delta(q^2-m^2)\, G_4(p,-p,q,-q) + \ldots$$

$\mathcal{D}_T$ is free propagator at $T \neq 0$ (calculation of $\operatorname{Sp} \ln \mathcal{D}_T$ see, e.g. in [6]). Expanding $\text{б}(E_p)$ at $\beta \to \infty$ we get finally the desired low-temperature expansion. The above expansion, we believe, is interesting and will be useful.

One of the immediate consequences of that is the confirmation of the fact, that the zero-temperature counterterms renormalize theory also at $T \neq 0$.

It will be interesting and important to generalize this formalism for the calculation of the effective action at finite temperature.

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APPENDIX

Here we prove the equation used in the proof of (11):

$$\sum_{i=1}^{n} \theta(x_i)\varkappa_i \prod_{j\neq i} \theta(\varkappa_j(x_j-x_i)) - \prod_{i=1}^{n} \theta(\varkappa_i x_i) = 0 \qquad \text{(A1)}$$

$\varkappa_i = \pm 1$, but not all of them are equal to (-1).

The poles of $\theta(z)$ lie in the points $z = 2n\pi i/\beta$. In the left-hand side they all cancel and, consequently, this analytical function is constant. At $X \to -\infty$ it tends to zero thus being zero everywhere.

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