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ЦЕНТРАЛЬНЫЙ НАУЧНО-ИССЛЕДОВАТЕЛЬСКИЙ ИНСТИТУТ
ИНФОРМАЦИИ И ТЕХНИКО-ЭКОНОМИЧЕСКИХ ИССЛЕДОВАНИЙ
ПО АТОМНОЙ НАУКЕ И ТЕХНИКЕ

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YANG-MILLS QUANTUM MECHANICS

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YANG-MILLS QUANTUM MECHANICS

Quantum mechanical properties of the Yang-Mills space-homogeneous model are considered in the Schrodinger representation. By means of compact variables the dependence of the wave function on "rotational" degrees of freedom is separated and effective Hamiltonians are obtained for "vibrational" degrees of freedom at various values of the moment. The effective Hamiltonian for $S = 0$ formally corresponds to the fundamental subsystem of the Yang-Mills classical mechanics and essentially differs from it by an additional singular potential.

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КВАНТОВАЯ МЕХАНИКА ЯНГА-МИЛЛСА

В работе рассмотрены кванто-механические свойства пространственно-однородной модели Янга-Миллса в представлении Шредингера. С помощью компактных переменных выделена зависимость волновой функции от "вращательных" степеней свободы и получены эффективные гамильтонианы для "колебательных" степеней свободы при различных значениях момента. Эффективный гамильтониан для $S = 0$ формально соответствует фундаментальной подсистеме классической механики Янга-Миллса и существенно отличается от неё дополнительным сингулярным потенциалом.

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1. Introduction

Despite the huge progress in theory of fundamental interactions based on the concept of local gauge invariance, low-energy consequences of the theory, in particular, spectral ones are obscure.

At present there exist several novel approaches to the problems of quark confinement and vacuum structure, such as, e.g. the quantum inverse method and the Monte Carlo numerical calculation.

Several years ago Feynman [1] has made an attempt to understand the low-energy behavior of theory using the traditional Schrodinger representation. He had made some simplifying assumptions, first of all there were no quarks in the model, SU(2) was the gauge group and, finally, the space had a dimension 2 + 1.

In the present paper we too shall make use of the Schrodinger representation, the gauge group is taken equal to SU(2), quarks in the model are lacking but instead of the last assump+

tion we assume that dynamical variables are independent of space coordinates.

The last assumption apparently corresponds to the "long-wave" limit of the theory, and to be correct, one should further take into account "short-wave" models. In this paper, however, we shall not take them into account but concentrate our attention on the solution of this space-homogeneous model. Its study seems to be a natural stage in the solution of the complete theory.

The above model in the classical (formal) limit $\hbar \to 0$ represents a classical gauge theory where the vector potential A depends on time only. This discrete mechanical system (Yang-Mills classical mechanics - YMCM)[2,3] is nonintegrable [2] and has strong statistical properties [4] (see also [5,6,7]).

The qualitative investigation of the space-homogeneous model energy spectrum has shown that there is no degeneracy in the spectrum and there occurs a "repulsion" of levels [8] that is expressed in the fact that the probability to find two neighboring levels at the distance Δ smaller than the average distance $\langle \Delta \rangle$, tends to zero as Δ^{β} , where β is the critical index.

In the present paper we investigate the space-homogeneous model in the Schrodinger representation. In the second section which is somewhat introductory, the necessary information from the quantum theory of gauge fields that we shall use later on is presented. In the third section basic relations of the space-homogeneous model are formulated. In the fourth - compact variables are introduced that substantially simplify the solution of the Schrodinger equation. In the fifth section we present the extraction of that part of the wave function that depends on the compact variables and, besides, effective Hamiltonians for invariant variables at different values of the total momentum as well as universal relations between them are obtained. In the sixth section we present the classification of levels and selection rules which they should satisfy. In conclusion we discuss the results obtained.

2. Hamiltonian formulation of the Yang-Mills field

Consider the Yang-Mills (YM) field with the group SU(2). In the gauge $A_0^a = 0$ the Hamiltonian has the form [9]

$$H = \int d^3\bar{r} \left[\frac{g^2}{2} E_k^a E_k^a + \frac{1}{2g^2} H_k^a H_k^a \right] , \tag{2.1}$$

where

$$H_k^a = \frac{1}{2} \varepsilon_{klm} F_{lm}^a , \quad F_{lm}^a = \partial_m A_l^a - \partial_l A_m^a + \varepsilon^{abc} A_l^b A_m^c \tag{2.2}$$

(i,j,k,l,m = 1,2,3; a,b,c = 1,2,3). As canonical variables serve the fields A_i^a, E_i^a

$$\left\{ E_k^a(\bar{r}), A_l^b(\bar{r}') \right\} = \delta^{ab} \delta_{kl} \delta(\bar{r}-\bar{r}') . \tag{2.3}$$

One should add to the classical equations of motion

$$\begin{aligned} \partial_0 E_k^a &= -\frac{\delta H}{\delta A_k^a} = \left\{ H, E_k^a \right\} , \\ \partial_0 A_k^a &= \frac{\delta H}{\delta E_k^a} = \left\{ H, A_k^a \right\} \end{aligned} \tag{2.4}$$

the coupling equations

$$G^a(\bar{r}) = \partial_i E_i^a - \varepsilon^{abc} A_i^b E_i^c = 0 \;, \tag{2.5}$$

that satisfy the following canonical commutation relations:

$$\{G^a(\bar{r}), A_k^b(\bar{r}')\} = (\delta^{ab}\partial_k - \varepsilon^{acb} A_k^c)\; \delta(\bar{r}-\bar{r}') \;, \tag{2.6}$$

$$\{G^a(\bar{r}), E_k^b(\bar{r}')\} = \varepsilon^{abc} E_k^c\; \delta(\bar{r}-\bar{r}') \;, \tag{2.7}$$

$$\{G^a(\bar{r}), G^b(\bar{r}')\} = \varepsilon^{abc} G^c(\bar{r})\, \delta(\bar{r}-\bar{r}') \;, \tag{2.8}$$

$$\partial_0 G^a = \{H, G^a\} = 0 \;. \tag{2.9}$$

From these relations follows that couplings G^a are independent of the time and are generators of infinitesimal gauge transformations that have remained after imposing the gauge condition $A_0^a = 0$. A and E are then transformed as follows:

$$\delta E_k^a(\bar{r}) = \{G(\alpha), E_k^a(\bar{r})\} = \varepsilon^{abc} E_k^b(\bar{r})\alpha^c(\bar{r}) \;, \tag{2.10}$$

$$\delta A_k^a(\bar{r}) = \{G(\alpha), A_k^a(\bar{r})\} = -\partial_k \alpha^a(\bar{r}) + \varepsilon^{abc} A_k^b(\bar{r})\alpha^c(\bar{r}) \;, \tag{2.11}$$

where

$$G(\alpha) = \int G^a(\bar{r})\, \alpha^a(\bar{r})\, d^3\bar{r}$$

and $\alpha^a(\bar{r})$ are the parameters of infinitesimal transformations.

On quantization dynamical variables, the Hamiltonian and couplings turn into operators satisfying simultaneous commutation relations

$$\hat{A}_i^a = A_i^a \quad , \quad \hat{E}_i^a = \frac{1}{i}\frac{\delta}{\delta A_i^a}$$

$$\left[\hat{E}_i^a(\bar{r}), A_j^b(\bar{r}')\right] = i\,\delta^{ab}\delta_{ij}\,\delta(\bar{r}-\bar{r}') \tag{2.12}$$

$$\left[\hat{G}^a(\bar{r}), \hat{A}_j^b(\bar{r}')\right] = \frac{1}{i}\left(\partial_j\delta^{ab} - \varepsilon^{acb}A_j^c\right)\delta(\bar{r}-\bar{r}') \tag{2.13}$$

$$\left[\hat{G}^a(\bar{r}), \hat{E}_j^b(\bar{r}')\right] = \frac{1}{i}\,\varepsilon^{abc}E_j^c\,\delta(\bar{r}-\bar{r}') \tag{2.14}$$

$$\left[\hat{G}^a(\bar{r}), \hat{G}^b(\bar{r}')\right] = \frac{1}{i}\,\varepsilon^{abc}\hat{G}^c(\bar{r})\,\delta(\bar{r}-\bar{r}') \tag{2.15}$$

$$\partial_0\hat{G}^a = \left[H, G^a\right] = 0 \tag{2.16}$$

The coupling equation (2.5) should be replaced by the condition on the wave function:

$$\hat{G}^a\Psi[A] = 0 \tag{2.17}$$

since $\hat{G}^a$ does not commute with canonical variables (2.13-14) and the operator equation $\hat{G}^a = 0$ should have brought to a contradiction. The fact that $\hat{G}^a$ commutes with H (2.16) implies that, as in the classical theory, $\hat{G}^a$ is independent of time. and hence H and $\Psi[A]$ are independent of definite sets of dynamical variables $\mathcal{A}_k^a$.

Thus the wave function of stationary states satisfies the Schrodinger equation

$$\int d^3r \left[-\frac{g^2}{2} \frac{\delta^2}{\delta \mathcal{A}_i^a \delta \mathcal{A}_i^a} + \frac{1}{2g^2} H_i^a H_i^a \right] \Psi[A] = \mathcal{E} \Psi[A] \, , \quad (2.18)$$

and coupling equations (2.17)

$$\left(\delta^{ab} \partial_k - \varepsilon^{abc} \mathcal{A}_k^b \right) \frac{1}{i} \frac{\delta}{\delta \mathcal{A}_k^c} \Psi[A] = 0 \, . \quad (2.19)$$

When classifying states of the gluon field we should need the expression for the total angular momentum:

$$\hat{m} = \int d^3 \bar{r} \left[\vec{r} \times \left[\vec{E}^a \times \vec{H}^a \right] \right] \quad (2.20)$$

that is composed of quatization of two summands, orbital and spin moments.

3. Space-homogeneous model

Let dynamical variables be independent of space coordinates, i.e. fields are spatially homogeneous. Then the Schrodinger equation (2.18) and the coupling equation (2.19) will be rewritten as

$$\left[-\frac{g^2}{2} \frac{\partial^2}{\partial \mathcal{A}_i^a \partial \mathcal{A}_i^a} + \frac{1}{4g^2} \left((\mathcal{A}_i^a \mathcal{A}_i^a)^2 - (\mathcal{A}_i^a \mathcal{A}_j^a)^2 \right) \right] \psi = \varepsilon \psi \, , \quad (3.1)$$

$$\hat{h}^a \psi = \frac{1}{i} \varepsilon^{abc} \mathcal{A}_i^b \frac{\partial}{\partial \mathcal{A}_i^c} \psi = 0 \quad (3.2)$$

and the density operator of the spin momentum, which as before will be denoted by $\hat{m}_i$, equals

$$\hat{m}_i = \frac{1}{i} \varepsilon_{ijk} A_j^a \frac{\partial}{\partial A_k^a} \qquad (3.3)$$

Both operators $\hat{m}_i$ and $\hat{n}^a$ (the second operator has the meaning of the isotopic spin density) commute with the Hamiltonian of the system (3.3) due to its invariance with respect to SO(3) × SO(3).

The complete set of commutation relations of the system (3.1) has the form

$$\begin{aligned}
&[H, \hat{n}^a] = [H, \hat{n}^a \hat{n}^a] = 0 \ , [H, \hat{m}_i] = [H, \hat{m}_i \hat{m}_i] = 0 , \\
&[\hat{n}^a, A_k^b] = i \varepsilon^{abc} A_k^c \qquad , [\hat{m}_i, A_j^b] = i \varepsilon_{ijk} A_j^b \\
&[\hat{n}^a, \hat{E}_k^b] = i \varepsilon^{abc} \hat{E}_k^c \quad , [\hat{m}_i, \hat{E}_k^b] = i \varepsilon_{ikj} \hat{E}_j^b \\
&[\hat{n}^a, \hat{n}^b] = i \varepsilon^{abc} \hat{n}^c \quad , [\hat{m}_i, \hat{m}_j] = i \varepsilon_{ijk} \hat{m}_k \\
&\qquad [\hat{n}^a, \hat{m}_i] = 0 , \quad [H, \hat{I}] = 0
\end{aligned} \qquad (3.4)$$

where $\hat{I}$ is the inversion operator. Commutation relations (3.4) have the form of common commutation relations for moments. Coupling equations(3.2) mean in this context that solutions with the isotopic spin density equal to zero - " S " states correspond to physical states.

Below we investigate the quantum-mechanical system (3.1-2) by means of compact variables introduced in [10] with this part of results being valid for the complete system (2.18-19) as well.

4. Compact variables

It is especially convenient to solve the Schrodinger equation (3.1) by means of compact variables introduced in [10]*. Let us represent the field matrix A^a_i in the form

$$A^a_i = O_1^{ab}(\varphi)\, E^b_k\, O^T_{2\,ki}(\theta) \quad , \tag{4.1}$$

where O_1 and O_2 are the orthogonal, and E the diagonal matrices parametrized by means of the Euler angles θ_i, φ_a and variables x, y, z , respectively. The wave function depends now on these nine variables

$$\Psi[A] = \Psi[x, y, z, \theta, \varphi] \tag{4.2}$$

and the task is to calculate the Laplacian in (2.18) in new variables.

It is known that in curvilinear coordinates the Laplacian has the following form:

$$\sum_{\mu,\nu} G^{-1/2} \frac{\partial}{\partial q_\mu}\left[G^{1/2} g^{-1}_{\mu\nu}\right]\frac{\partial}{\partial q_\nu} \tag{4.3}$$

where $g_{\mu\nu}$ is the metric tensor

$$ds^2 = g_{\mu\nu}(q)\, dq_\mu dq_\nu \tag{4.4}$$

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* The author is indebted to L.D.Fadeev for attracting his attention to [11,12] where similar variables were introduced in connection with the problem of the Gribov ambiguities.

G is the determinant of the matrix $g_{\mu\nu}$, and $g^{-1}_{\mu\nu}$ denotes a matrix inverse to the matrix $g_{\mu\nu}$. The volume element is equal to

$$d\tau = |G^{1/2}|\, dq_1 \cdots dq_N \quad . \tag{4.5}$$

In old variables the interval reads

$$ds^2 = \sum_{a,i} dA_i^a\, dA_i^a = 2T(dt)^2 , \tag{4.6}$$

where T is the kinetic energy and, hence, $g_{\mu\nu} = \delta_{ij}\,\delta^{ab}$. In order to obtain ds^2 and $g_{\mu\nu}$ in new variables, let us substitute (4.1) in (4.6). As a result we obtain

$$ds^2 = dx^2 + dy^2 + dz^2 + \sum_{a=1}^{3} I_a (d\Omega_a^2 + d\omega_a^2) - 2J_a\, d\Omega_a\, d\omega_a \tag{4.7}$$

where

$$I_1 = y^2 + z^2 , \quad I_2 = x^2 + z^2 , \quad I_3 = x^2 + y^2$$

$$J_1 = 2yz , \quad J_2 = 2xz , \quad J_3 = 2xy , \tag{4.8}$$

and angular velocities Ω and ω are expressed via matrices O_1 and O_2 by relations

$$d\omega = O_1^T dO_1 , \qquad d\Omega = O_2^T dO_2 ,$$

$$d\omega_a = -\frac{1}{2}\varepsilon_{abc}\, d\omega_{bc} , \quad d\Omega_i = -\frac{1}{2}\varepsilon_{ijk}\, d\Omega_{jk} . \tag{4.9}$$

The relation (4.7) has been obtained in [10] and required no further transformations when investigating the classical dynamics of the space-homogeneous model [4], since using it one already may obtain equations of motion for Ω and ω. However, during quantization the angular velocities Ω and ω

should be expressed via Euler angles θ_a and φ_a in order to obtain a new metric tensor (4.4).

The expression of the angular velocity via Euler angles is well-known from the course of the solid body mechanics [13]

$$d\Omega_a = V_{ab}(\theta)\, d\theta_b \quad , \quad d\omega_a = v^{\varphi}_{ab}(\varphi)\, d\varphi_b \tag{4.10}$$

where

$$V_{ab}(\theta) = v^{\varphi}_{ab}(\varphi \to \theta) = \begin{Vmatrix} -\sin\theta_2 , & \cos\theta_3 \sin\theta_3 & 0 \\ \sin\theta_2 , & \sin\theta_3 \cos\theta_3 & 0 \\ \cos\theta_2 , & 0 & 1 \end{Vmatrix} \tag{4.11}$$

Substituting (4.10) in (4.7) we obtain:

$$ds^2 = dx^2 + dy^2 + dz^2 + \sum_{a=1}^{3} I_a \sum_{b,c=1}^{3} (V_{ab} V_{ac}\, d\theta_b d\theta_c + v^{\varphi}_{ab} v^{\varphi}_{ac}\, d\varphi_b d\varphi_c) - 2 \sum_{a=1}^{3} J_a \sum_{b,c=1}^{3} V_{ab}\, v^{\varphi}_{ac}\, d\theta_b\, d\varphi_c \tag{4.12}$$

Let us introduce the following notations:

$$\begin{gathered} q_1 = x , \quad q_2 = y , \quad q_3 = z \\ q_4 = \theta_1 , \quad q_5 = \theta_2 , \quad q_6 = \theta_3 \\ q_7 = \varphi_1 , \quad q_8 = \varphi_2 , \quad q_9 = \varphi_3 \end{gathered} \tag{4.13}$$

and make (4.12) equal to (4.4), then

$$\begin{gathered} g_{11} = 1 , \; g_{22} = 1 , \; g_{33} = 1 , \; g_{14} = \cdots = g_{19} = 0 \\ g_{44} = \sum_{a=1}^{3} I_a V_{a4} V_{a4} , \; \ldots \qquad g_{49} = -\sum_{a=1}^{3} J_a V_{a4} v^{\varphi}_{a9} , \; \ldots \end{gathered} \tag{4.14}$$

etc. Thus, the matrix $\widetilde{g_{\mu\nu}}$ has the dimensions $\| 9 \times 9 \|$ and is composed of a unit matrix $\| 3 \times 3 \|$ in the upper left angle, a matrix $\| 6 \times 6 \|$ in the lower right angle, and zeros in the remaining places. In turn, the matrix $\| 6 \times 6 \|$ is composed of four matrices $\| 3 \times 3 \|$

$$\left\| \begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline C & D \end{array} \right\| \tag{4.15}$$

that may be obtained substituting (4.11) in (4.14)

$$\mathcal{A} = \left\| \begin{array}{ccc} \begin{array}{l} I_1 \sin^2\theta_2 \cos^2\theta_3 \\ I_2 \sin^2\theta_2 \sin^2\theta_3 \\ I_3 \cos^2\theta_2 \end{array} , & (I_2 - I_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 , & I_3 \cos\theta_2 \\ (I_2 - I_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 , & I_1 \sin^2\theta_3 + I_2 \cos^2\theta_3 , & 0 \\ I_3 \cos\theta_2 , & 0 , & I_3 \end{array} \right\| \tag{4.16}$$

$$\mathcal{B} = \left\| \begin{array}{ccc} \begin{array}{l} -J_1 \sin\theta_2 \cos\theta_3 \sin\varphi_2 \cos\varphi_3 \\ -J_2 \sin\theta_2 \sin\theta_3 \sin\varphi_2 \sin\varphi_3 \\ -J_3 \cos\theta_2 \cos\varphi_2 \end{array} , & \begin{array}{l} J_1 \sin\theta_2 \cos\theta_3 \sin\varphi_3 \\ -J_2 \sin\theta_2 \sin\theta_3 \cos\varphi_3 \end{array} , & -J_3 \cos\theta_2 \\ \begin{array}{l} J_1 \sin\theta_3 \sin\varphi_2 \cos\varphi_3 \\ -J_2 \cos\theta_3 \sin\varphi_2 \sin\varphi_3 \end{array} , & \begin{array}{l} -J_1 \sin\theta_3 \sin\varphi_3 \\ -J_2 \cos\theta_3 \cos\varphi_3 \end{array} , & 0 \\ -J_3 \cos\varphi_2 , & 0 , & J_3 \end{array} \right\| \tag{4.17}$$

matrix $C = B^{T}$, and $\mathcal{D}$ is obtained from $\mathcal{A}$ at the replacement $\theta_a \to \varphi_a$.

Let us calculate the metric tensor $g_{\mu\nu}$ determinant in (4.3) which is equal to the determinant of the matrices (4.15). For that purpose let us make the following operations: multiply the third column by $\cos\theta_2$ and subtract it from the first column, multiply the sixth column by $\cos\varphi_2$ and subtract it from the fourth column, multiply the third column by J_3/I_3 and add it to the sixth. The obtained determinant expand first by the sixth column, then by the third line. We then obtain:

$$G = \frac{(I_3^2 - J_3^2)}{I_3} I_3 \cdot \operatorname{Det} \| 4 \times 4 \| \quad (4.18)$$

The first line of the matrix $\| 4 \times 4 \|$ determinant is proportionate to $\sin\theta_2$, the first column - to $\sin\theta_2$, the third line - to $\sin\varphi_2$, the third column - to $\sin\varphi_2$, so

$$G = (I_3^2 - J_3^2) \sin^2\theta_2 \sin^2\varphi_2 \operatorname{Det} \| X \| \quad (4.19)$$

The matrix $\| X \|$ is presented in Appendix A, and its determinant is equal to $(I_1^2 - J_1^2) \cdot (I_2^2 - J_2^2)$. Finally we obtain:

$$G = (I_1^2 - J_1^2)(I_2^2 - J_2^2)(I_3^2 - J_3^2) \sin^2\theta_2 \sin^2\varphi_2 \quad . \quad (4.20)$$

The volume element is written as

$$d\tau = \sin\theta_2 \prod_i d\theta_i \sin\varphi_2 \prod_a \varphi_a \, D(x,y,z) \, dx \, dy \, dz \; , \quad (4.21)$$

$$D = |x^2 - y^2| \, |y^2 - z^2| \, |z^2 - x^2|$$

(4.22)

It is seen from (4.21) that $d\tau$ is proportionate to the product of invariant dimensions by the group of orthogonal matrices.

After some simple calculations, one may calculate the inverse matrix $g^{-1}_{\mu\nu}$ and thus the kinetic energy operator:

$$g^2\hat{T} = -\frac{1}{2}\sum_{\mu,\nu=1}^{3} D^{-1}\frac{\partial}{\partial q_\mu} D \frac{\partial}{\partial q_\nu} -$$
$$-\frac{1}{2}\sum_{\mu\nu=4}^{9} \sin^{-1}\theta_2 \sin^{-1}\varphi_2 \frac{\partial}{\partial q_\mu} \sin\theta_2 \sin\varphi_2 \, g^{-1}_{\mu\nu} \frac{\partial}{\partial q_\nu} = \tag{4.23}$$
$$= -\frac{1}{2}\left(\partial_x^2+\partial_y^2+\partial_z^2\right) - \frac{1}{2}\left(\partial_x \ln D\,\partial_x + \partial_y \ln D\,\partial_y + \partial_z \ln D\,\partial_z\right)$$
$$+\frac{1}{2}\sum_{a=1}^{3}\frac{I_a(\hat{M}_a^2+\hat{N}_a^2) + 2J_a\hat{M}_a\hat{N}_a}{I_a^2 - J_a^2},$$

where

$$M_1 = \frac{1}{i}\left\{-\frac{\cos\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \sin\theta_3\frac{\partial}{\partial\theta_2} + \mathrm{ctg}\,\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\right\}$$
$$M_2 = \frac{1}{i}\left\{\frac{\sin\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \cos\theta_3\frac{\partial}{\partial\theta_2} - \mathrm{ctg}\,\theta_2\sin\theta_3\frac{\partial}{\partial\theta_3}\right\} \tag{4.24}$$
$$M_3 = \frac{1}{i}\left\{\frac{\partial}{\partial\theta_3}\right\}$$

and operators $\hat{N}_a$ are obtained from (4.24) at replacement $\theta\to\varphi$

Operators (4.24) satisfy the following commutation relations:

$$[\hat{M}_a, \hat{M}_b] = -i\varepsilon_{abc}\hat{M}_c, \quad [\hat{N}_a, \hat{N}_b] = -i\varepsilon_{abc}\hat{N}_c$$
$$[\hat{M}_a, \hat{N}_b] = 0 . \tag{4.25}$$

These commutation relations differ from common ones for moments by the minus sign on the right hand side. This difference will be accounted for below.

The expression (4.23) for the kinetic energy operator has an obvious physical interpretation based on the analogy between the above system and solid body mechanics. As has been noted in [10], the classical equations of YM in the variables (4.1) describe the rotation of the "gauge" body in internal and ordinary spaces, the variables x, y, z describe its own "vibrations", since inertia moments I_a, J_a (4.7) depend on x, y, z and $\hat{M}_a$ and $\hat{N}_a$ are the projections of angular momenta $\hat{m}$ and $\hat{n}$ on the "moving" coordinate system. It is convenient to use this analogy in the quantum case as well, **assuming** that $\hat{T}$ in (4.23) is composed of two summands describing the vibrational and rotational energies of the gluon field.

In Appendix B expressions are obtained for the angular momentum operator in the "fixed" coordinate system via Euler angles

$$
\begin{aligned}
\hat{m}_1 &= \frac{1}{i}\left\{\cos\theta_1 \, ctg\,\theta_2 \frac{\partial}{\partial\theta_1} - \sin\theta_1 \frac{\partial}{\partial\theta_2} + \frac{\cos\theta_1}{\sin\theta_2}\frac{\partial}{\partial\theta_3}\right\} \\
\hat{m}_2 &= \frac{1}{i}\left\{-\sin\theta_1 \, ctg\,\theta_2 \frac{\partial}{\partial\theta_1} + \cos\theta_1 \frac{\partial}{\partial\theta_2} + \frac{\sin\theta_1}{\sin\theta_2}\frac{\partial}{\partial\theta_3}\right\} \qquad (4.26) \\
\hat{m}_3 &= \frac{1}{i}\left\{\frac{\partial}{\partial\theta_1}\right\}
\end{aligned}
$$

Operators $\hat{n}_a$ are obtained by the replacement $\theta_a \to \varphi_a$. Operators $\hat{m}$ and $\hat{n}$ satisfy the ordinary commutation relations (3.4).

Now the change in the sign of commutators (4.25) as compared to (3.4) is clear. It is determined, as is first noted by O.Klein in the molecule theory [14], by the fact that components $\hat{m}$ and $\hat{n}$ (4.26) are defined with respect to fixed axes, i.e. angular momentum operators $\hat{m}$ and $\hat{n}$ turn round the "fixed" coordinate system, whereas components $\hat{M}$ and $\hat{N}$ (4.24) defined with respect to the "moving" system, turn round the system with respect to which they are defined (see for details Appendix B). The total angular momentum operator expressed via Euler angles has the form:

$$\hat{m}^2 = \hat{M}^2 = \Big\{ -\frac{\partial^2}{\partial\theta_2^2} - ctg\,\theta_2 \frac{\partial}{\partial\theta_2} - \frac{1}{\sin^2\theta_2}\left(\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2}\right) + 2\,\frac{\sin^{-2}\theta_2}{\cos^{-1}\theta_2}\,\frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big\} \tag{4.27}$$

Thus, the Schrodinger equation (3.1) of the space-homogeneous model in the presented compact variables has the form:

$$\frac{g^2}{2}\Big[-\partial_x^2-\partial_y^2-\partial_z^2-\partial_x \ln D\,\partial_x-\partial_y \ln D\,\partial_y-\partial_z \ln D\,\partial_z+ \sum_{a=1}^{3}\frac{I_a(\hat{M}_a^2+\hat{N}_a^2)+2J_a\hat{M}_a\hat{N}_a}{I_a^2-J_a^2}\Big]\psi+\frac{1}{2g^2}(x^2y^2+y^2z^2+z^2x^2)\psi=\varepsilon\psi \tag{4.28}$$

The integration measure is given by the formula (4.21), commutation relations - by the formulae (4.25),(3.4) and coupling equations (3.2) imply that physical states have an isospin $\hat{n}_a^2\psi = 0$ equal to zero.

5. Diagonalization of the Hamiltonian by rotational degrees of freedom

Let us take as basic functions the normalized eigenfunctions of the operators

$$\hat{m}^2 = \hat{M}^2, \quad \hat{m}_3, \quad \hat{M}_3,$$
$$\hat{n}^2 = \hat{N}^2, \quad \hat{n}_3, \quad \hat{N}_3. \tag{5.1}$$

and denote them via $|S\, S_3\, S_3'\rangle$ and $|j\, j_3\, j_3'\rangle$, respectively

$$\hat{m}^2 |S S_3 S_3'\rangle = S(S+1)\, |S S_3 S_3'\rangle,$$
$$\hat{m}_3 |S S_3 S_3'\rangle = S_3\, |S S_3 S_3'\rangle, \tag{5.2}$$
$$\hat{M}_3 |S S_3 S_3'\rangle = S_3'\, |S S_3 S_3'\rangle,$$

and similarly for $\hat{n}^2, \hat{n}_3, \hat{N}_3$. For the given representation S is fixed and takes the value

$$0,\ 1/2,\ 1,\ 3/2,\ \dots \tag{5.3}$$

and for each S there exist $(2S+1)^2$ states with

$$S_3 = S, S-1, \dots -S+1, -S,$$
$$S_3' = S, S-1, \dots -S+1, -S, \tag{5.4}$$

i.e. we have a $(2S+1)^2$ dimension representation of operators $\hat{m}^2, \hat{m}_3$ and $\hat{M}_3$. The same is for $\hat{n}^2, \hat{n}_3, \hat{N}_3$. Matrix elements of the operators $\hat{m}_1, \hat{M}_1, \hat{m}_2$ and $\hat{M}_2$ in this basis are presented in Appendix B.

Eigenfunctions of moments $|S, S_3 S_3'\rangle$ are expressed via complex conjugation of D-function [15]:

$$D^S_{S_3 S_3'}(\theta_1 \theta_2 \theta_3) = +e^{-iS_3\theta_1}\, d^S_{S_3 S_3'}(\theta_2)\, e^{-iS_3'\theta_3} \tag{5.5}$$

satisfying equation

$$\left\{ -\frac{\partial^2}{\partial\theta_2^2} - ctg\,\theta_2 \frac{\partial}{\partial\theta_2} - \sin^{-2}\theta_2 \left(\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_3^2} \right) + \right.$$

$$\left. + 2\,\frac{\cos\theta_2}{\sin^2\theta_2} \frac{\partial^2}{\partial\theta_1\partial\theta_3} \right\} D^s_{s_3 s_3'}(\theta_1\theta_2\theta_3) = s(s+1)\, D^s_{s_3 s_3'}(\theta_1\theta_2\theta_3)$$

where the function $d^s_{s_3 s_3'}(\theta_2)$ is presented in Appendix B,

$$|s\, s_3\, s_3'\rangle = D^{*s}_{s_3 s_3'} = (-1)^{s_3 - s_3'} D^s_{-s_3 - s_3'} \qquad (5.6)$$

and is normalized by equation

$$\int D^{*s}_{s_3 s_3'} D^{\ell}_{\ell_3 \ell_3'} d\mu = \frac{8\pi}{2s+1}\, \delta^{s\ell} \delta_{s_3 \ell_3} \delta_{s_3' \ell_3'} \qquad (5.7)$$

where $d\mu$ is the invariant measure on the orthogonal matrix group.

We have already noted that coupling constants (3.2) are equivalent to the requirement $j = j_3 = j_3' = 0$, therefore, the wave function is independent of the variables φ_a , which is consistent with the general statement of the second section (see (2.17) and below).

$$\psi_{j=0}(x, y, z, \theta, \varphi) = \frac{1}{(8\pi^2)^{1/2}}\, \psi(x, y, z, \theta) \qquad (5.8)$$

In (4.28) the dependence of the Hamiltonian on compact variables is now determined only by the summand:

$$g^2 \hat{T} = \frac{1}{2} \sum_{a=1}^{3} \frac{I_a}{I_a^2 - J_a^2}\, \hat{M}_a^2 \qquad (5.9)$$

that coincides with the Hamiltonian of the asymmetrical top [15]. In states with definite S, S_3, S_3' the Hamiltonian is not diagonal, therefore, one should choose a wave function in the form of a superposition:

$$\psi^{SS_3}(x,y,z,\theta) = \sum_{S_3'=-S}^{S} \Phi^{SS_3}_{S_3'}(x,y,z)\,|SS_3S_3'\rangle \tag{5.10}$$

Since only matrix elements $\hat{T}$ with $\Delta S_3' = 0, \pm 2$ are different from zero, the secular equation of the degree $(2S+1)$ decays into two independent equations of degree S and $S+1$ (see Appendix B) whose solution for lower spins reads:

$$S = 0 \qquad , \qquad T(0) = 0 \quad , \tag{5.11a}$$

$$S = 1/2 \qquad , \qquad T_A(\tfrac{1}{2}) = \frac{1}{8}\sum_{a=1}^{3}\frac{I_a}{I_a^2 - J_a^2} \quad , \tag{5.11b}$$

$$S = 1 \qquad , \qquad T_{B_1}(1) = \frac{1}{2}\left[\frac{I_1}{I_1^2 - J_1^2} + \frac{I_2}{I_2^2 - J_2^2}\right] \quad ,$$

$$T_{B_2}(1) = \frac{1}{2}\left[\frac{I_1}{I_1^2 - J_1^2} + \frac{I_3}{I_3^2 - J_3^2}\right] \quad , \tag{5.11c}$$

$$T_{B_3}(1) = \frac{1}{2}\left[\frac{I_2}{I_2^2 - J_2^2} + \frac{I_3}{I_3^2 - J_3^2}\right] \quad ,$$

etc. Since matrix elements $\hat{T}$ depend on different sets $I_a / I_a^2 - J_a^2$, there exist between them universal relations, e.g.

$$T_{B_1}(1) + T_{B_2}(1) + T_{B_3}(1) = 8\, T_A(1/2) \tag{5.12}$$

Indices A, B_i denote the irreducible representations of the discrete group D_2 (see section 6).

Let us write out equations satisfied by "vibrational" degrees of freedom x, y, z for various values of spins. Let us denote via H_0 the operator

$$H_0 = -\frac{g^2}{2}\left[\Delta + \vec{\nabla} \ln D \vec{\nabla}\right] + \frac{1}{2g^2}\left[x^2y^2 + y^2z^2 + z^2x^2\right], \tag{5.13}$$

then we obtain for $S = 0$

$$H_0 \Phi_0^{00} = \varepsilon \Phi_0^{00} \tag{5.14}$$

for $S = ½$

$$\left[H_0 + \frac{1}{8}\left(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{x^2+z^2}{(x^2-z^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2}\right)\right]\Phi_{\pm 1/2}^{1/2\, S_3} = \varepsilon\, \Phi_{\pm 1/2}^{1/2\, S_3} \tag{5.15}$$

etc. The system (5.14) is the quantum analog of the classical fundamental subsystem introduced in [4] and it essentially differs from the latter by an additional summand that contains first derivatives by the variables x, y, z. This additional potential may be simplified by introducing another normalization of wave function. Indeed, the volume element has been so far equal to (4.21), and now let us introduce the wave function

$$\tilde{\psi} = D^{1/2}\psi \tag{5.16}$$

that is normalized by the simple volume element

$$d\tau = dx\, dy\, dz\, \sin\theta_2 \prod_a d\theta_a \sin\varphi_2 \prod_a d\varphi_a \tag{5.17}$$

One then should properly transform differential operators. In particular, H_0 is equal to

$$H_0 = -\frac{g^2}{2}\left[\Delta + (1/4D^2)(\vec{\nabla}D)^2 - (1/2D)\Delta D\right] + 1/2g^2\left(x^2y^2 + y^2z^2 + z^2x^2\right) \tag{5.18}$$

and contains the additional potential

$$g^2 U_{eff} = \frac{1}{4D}\,\Delta D - \frac{1}{8D^2}\left(\vec{\nabla} D\right)^2 \tag{5.19}$$

where D is equal to (4.22).

6. Classification of states and selection rules

The Hamiltonian (4.28) and commutation rules (4.25) have a peculiar symmetry, e.g. they are invariant with respect to transformations

$$\begin{array}{lllll} M_1 \to -M_1 & N_1 \to N_1 & I_1 \to I_1 & J_1 \to -J_1 & x \to -x \\ M_2 \to M_2 & N_2 \to N_2 & I_2 \to I_2 & J_2 \to J_2 & y \to y \\ M_3 \to -M_3 & N_3 \to N_3 & I_3 \to I_3 & J_3 \to -J_3 & z \to -z \end{array} \tag{6.1a}$$

$$\begin{array}{lllll} M_1 \to -M_1 & N_1 \to -N_1 & I_1 \to I_1 & J_1 \to J_1 & x \to x \\ M_2 \to M_2 & N_2 \to N_2 & I_2 \to I_2 & J_2 \to J_2 & y \to y \\ M_3 \to -M_3 & N_3 \to -N_3 & I_3 \to I_3 & J_3 \to J_3 & z \to z \end{array} \tag{6.1b}$$

etc. This discrete symmetry arises due to ambiguities in the transformation (4.1). If we introduce unit matrices on the right and left of the matrix E in (4.1) and then represent them as the production of orthogonal matrices, we shall obtain

$$\mathcal{A} = O_1 E O_2^T = O_1 R^T R E P^T P O_2^T = O_1' E' O_2'^T \tag{6.2}$$

where

$$O_1' = O_1 R^T, \quad O_2' = O_2 P^T, \quad E' = R E P^T. \qquad (6.3)$$ *

The only requirement which orthogonal matrices R and P should satisfy is the diagonality of E', since O_1' and O_2' are orthogonal. It is easy to see that this is connected with the choice of axes of a three-dimensional coordinate system. There exist 24 methods for the choice of right coordinate system and as much for the left one. One may choose three independent orthogonal matrices R_1 R_2 R_3 equal, respectively, to

$$R_1 = \left\| \begin{matrix} -1 & & 0 \\ & 1 & \\ 0 & & -1 \end{matrix} \right\| \quad R_2 = \left\| \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \right\| \quad R_3 = \left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \right\| \qquad (6.4)$$

and satisfying the condition:

$$R_1^2 = 1, \quad R_2^4 = 1, \quad R_3^3 = 1, \qquad (6.5)$$

so that the arbitrary matrix R of axis permutation is obtained by a subsequent multiplication of R_1, R_2 and R_3:

$$R_1^i R_2^k R_3^j \qquad (6.6)$$

where $i = 1,2;$ $k = 1,2,3,4;$ $j = 1,2,3$. One may obtain the matrix of left permutations by means of the matrix R_0:

$$R_0 = \left\| \begin{matrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{matrix} \right\|, \qquad Det\, R_0 = -1 \qquad (6.7)$$

Thus, there exist 48 x 48 possible matrices R and P but the requirement of E' being diagonal is satisfied only

*The author is indebted to H.M.Asatryan for attracting his attention to the relation (6.2-3).

by those pairs of R and P for which

$$R\,P^{T} = \begin{Vmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & \pm 1 \end{Vmatrix} \qquad (6.8)$$

The following matrices P are the solution of conditions (6.8) for three independent matrices R_i :

$$\begin{Vmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \end{Vmatrix} \qquad (6.9)$$

Let us expand these matrices (6.9) by the basic ones R_i (6.4) and we shall obtain for R_1 :

$$P = R_1;\ R_1^2;\ R_2^2;\ R_1 R_2^2;\ R_1 R_0;\ R_1^2 R_0;\ R_2^2 R_0;\ R_1 R_2^2 R_0 , \qquad (6.10)$$

for R_2

$$P = R_2;\ R_1 R_2;\ R_2 R_1;\ R_2^3, R_2 R_0, R_1 R_2 R_0, R_2 R_1 R_0, R_2^3 R_0 \qquad (6.11)$$

and, finally, for R_3

$$P = R_3;\ R_1 R_3;\ R_3 R_1;\ R_2^2 R_3;\ R_3 R_0;\ R_1 R_3 R_0;\ R_3 R_1 R_0;\ R_2^3 R_0 \qquad (6.12)$$

At permutations of R_1, R_2 and R_3 the Euler angles are transformed as follows:

$$R_1 \qquad \theta_1 \to \theta_1 + \pi, \quad \theta_2 \to \pi - \theta_2, \quad \theta_3 \to \pi - \theta_3 \qquad (6.13a)$$

$$R_2 \qquad \theta_1 \to \theta_1, \quad \theta_2 \to \theta_2, \quad \theta_3 \to \theta_3 + \pi/2 \qquad (6.13b)$$

$$R_3 \qquad \theta_1 \to \theta_1, \quad \theta_2 \to \theta_2 + \pi/2, \quad \theta_3 \to \theta_3 + \pi/2 \qquad (6.13c)$$

Consider separately the admissible pairs of transformations $R \times P$ (6.4) and (6.10-12). Let $R = R_1$, then angles

θ_a transform by (6.13a) and angles φ_a, in accord with (6.10), by means of eight combinations of (6.13a,b,c), with variables x, y, z transforming as

$$E \to E' \; ; \; R_1 \times P : \quad x' = \pm x, \; y' = \pm y, \; z' = \pm z . \qquad (6.14a)$$

At $R = R_2, R_3$ angles θ_a transform by (6.13b), (6.13c), and φ_a by combinations (6.11), (6.12), and x, y, z as follows:

$$E \to E' \; ; \; R_2 \times P : \quad x' = \pm y, \; y' = \pm x, \; z' = \pm z , \qquad (6.14b)$$

$$E \to E' \; ; \; R_3 \times P : \quad x' = \pm y, \; y' = \pm z, \; z' = \pm x \qquad (6.14c)$$

For example, in (6.1a) $R = R_1$, $P = R_1^2$ and in (6.1b) $R = R_1$, $P = R_1$ etc.

Due to the above ambiguity (6.2-3) there arise selection rules for equalities of the system (4.28) owing to the following: the wave function Ψ is an unambiguous function of the variables $\mathcal{A}$ and it should remain so after transition to new variables x, y, z, θ, φ too, i.e. it should be invariant with respect to the above transformations. However, if the Schrodinger equation is solved in new variables, one generally obtains solutions $\Psi(x, y, z, \theta, \varphi)$ noninvariant with respect to the transformations $R \times P$. Therefore, to provide the unambiguity of the wave function, one should symmetrize the function $\Psi(x, y, z, \theta, \varphi)$ with respect to the transformations $R \times P$.

Eigenfunctions of the moment $|S S_3 S_3'\rangle$ transform at the permutations R as follows:

$$\mathcal{R}_1 |S S_3 S_3'\rangle = e^{i\pi(S + S_3')} |S S_3 S_3'\rangle , \qquad (6.15a)$$

$$\mathcal{R}_2 |s s_3 s_3'\rangle = e^{i\frac{\pi}{2}s_3'} |s s s_3'\rangle \tag{6.15b}$$

$$\mathcal{R}_1 \mathcal{R}_2^2 |s s_3 s_3'\rangle = e^{i\pi s} |s s_3 - s_3'\rangle \tag{6.15c}$$

and act on $|j j_3 j_3'\rangle$ like $\mathcal{R}_i$. Relations (6.15) are easy to obtain using the fact that Euler angles at the permutations $\mathcal{R}_i$ transform by (6.13) and that $|s s_3 s_3'\rangle$ is equal to $\mathcal{D}^{*s}_{s_3 s_3'}$ (5.6).

The wave function with $j = 0$ (5.8) is invariant with respect to the action of operators $\mathcal{P}_i$ on the angles φ_a, therefore, one should symmetrize the wave function over the remaining variables. Under the action of $R \times P$ it transforms as follows:

$$\mathcal{R}_2 \times \mathcal{P}\psi^{s s_3}(x,y,z,\theta) = \sum_{s_3'=-s}^{s} \phi^{s s_3}_{s_3'}(\pm y, \pm x, \pm z) e^{i\frac{\pi}{2}s_3'} |s s_3 s_3'\rangle, \tag{6.16a}$$

$$\mathcal{R}_1 \times \mathcal{P}\psi^{s s_3}(x,y,z,\theta) = \sum_{s_3'=-s}^{s} \phi^{s s_3}_{s_3'}(\pm x, \pm y, \pm z) e^{i\pi(s+s_3')} |s s_3 - s_3'\rangle, \tag{6.16b}$$

$$\mathcal{R}_1 \mathcal{R}_2^2 \times \mathcal{P}\psi^{s s_3}(x,y,z,\theta) = \sum_{s_3'=-s}^{s} \phi^{s s_3}_{s_3'}(\pm x, \pm y, \pm z) e^{i\pi s} |s s_3 - s_3'\rangle, \tag{6.16c}$$

where $\mathcal{P}$ is for transformations satisfying (6.8), i.e. (6.10), (6.13a), (6.14a) and (6.11), (6.13b), (6.14b). In order to find out transformation properties of $\phi^{s s_3}_{s_3'}(x,y,z)$ one should solve equations of the type of (5.14-15). In the final analysis just their behavior will determine the selection rules.

7. Conclusion

Note in conclusion that by means of compact variables (4.1) one may explicitly solve the coupling equations also in the general case, and as a result summands $\sim 1/g$ emerge in the Hamiltonian [11,12]. This occurs due to the fact that at such a quantization the group of gauge transformations is treated exactly. On the other hand, the space-homogeneous model is the long-wave limit of the theory, therefore, one may assume that it is a zero approximation in the expansion by inverse powers g. In turn, the Schrodinger equation (2.18) describes the stationary levels of the gauge field, gluonium, therefore, the space-homogeneous model describes the gluonium spectrum in the zero approximation of the $1/g$ expansion.

Note also that the model considered, in principle, arises also in the gauge theories with the group SU(N) at $N \to \infty$ [16,17,18] as well as at stochastic quantization [19,22]. Finally, there recently appeared a new paper by Luscher [23] on the gluonium spectrum calculation on a torus, where the space-homogeneous model arose quite naturally. Therefore, the complete solution of this model may shed light on the understanding of the low-energy behavior of gauge theories.

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Appendix A

The matrix $\|X\|$ in (4.19) has the form

$$\left\| \begin{matrix} I_1 C_\theta^2 + I_2 S_\theta^2, & (I_2 - I_1) C_\theta S_\theta, & -J_1 C_\theta C_\varphi - J_2 S_\theta S_\varphi, & J_1 C_\theta S_\varphi - J_2 S_\theta C_\varphi \\ (I_2 - I_1) C_\theta S_\theta, & I_1 S_\theta^2 + I_2 C_\theta^2, & J_1 S_\theta C_\varphi - J_2 C_\theta S_\varphi, & -J_1 S_\theta S_\varphi - J_2 C_\theta C_\varphi \\ -J_1 C_\theta C_\varphi - J_2 S_\theta S_\varphi, & J_1 S_\theta C_\varphi - J_2 C_\theta S_\varphi, & I_1 C_\varphi^2 + I_2 S_\varphi^2, & (I_2 - I_1) C_\varphi S_\varphi \\ J_1 C_\theta S_\varphi - J_2 S_\theta C_\varphi, & -J_1 S_\theta S_\varphi - J_2 C_\theta C_\varphi, & (I_2 - I_1) C_\varphi S_\varphi, & I_1 S_\varphi^2 + I_2 C_\varphi^2 \end{matrix} \right\| \qquad \text{(A.1)}$$

where $C_\theta = \cos\theta_3, S_\theta = \sin\theta_3, C_\varphi = \cos\varphi_3, S_\varphi = \sin\varphi_3$.
Expanding the determinant of the matrix (A.1) in the second order minor determinants, we obtain after some simple calculations

$$\operatorname{Det}\|X\| = (I_1^2 - J_1^2)(I_2^2 - J_2^2) \qquad \text{(A.2)}$$

Appendix B

Let us denote the unit vectors of the "fixed" coordinate system in ordinary and isotopic spaces by $\vec{l}_a$ and $\vec{K}_a$, and in the "moving" coordinate system by $\vec{l}'_a$ and $\vec{K}'_a$. Let us also introduce unit vectors $\vec{e}_a^{\,(\theta,\varphi)}$ that indicate the direction of axes of rotations to the Euler angles θ_a and φ_a. Then the above rotations may be defined with respect both to "fixed" system $\vec{l}_a$ and $\vec{K}_a$ and "moving" system $\vec{l}'_a$ and $\vec{K}'_a$

$$\vec{e}_a^{\,(\theta)} = U_{ab} \vec{K}_b \quad , \qquad \vec{e}_a^{\,(\varphi)} = u_{ab} \vec{l}_b \,, \qquad \text{(B.1a)}$$

$$\vec{e}_a^{\,(\theta)} = V_{ab} \vec{K}'_b \quad , \qquad \vec{e}_a^{\,(\varphi)} = v_{ab} \vec{l}'_b \qquad \text{(B.1b)}$$

where

$$U_{ab}(\theta) = u_{ab}(\varphi \to \theta) = \left\| \begin{matrix} 0, & -\sin\theta_1, & \cos\theta_1 \sin\theta_2 \\ 0, & \cos\theta_1, & \sin\theta_1 \sin\theta_2 \\ 1, & 0, & \cos\theta_2 \end{matrix} \right\| \qquad \text{(B.2)}$$

and matrices V_{ab}, v_{ab} are given by the formulae (4.11).

From (B.1-2) we have

$$\vec{K}_a = U^{-1}_{ab} \vec{e}^{\,(\theta)}_b , \qquad \vec{l}_a = u^{-1}_{ab} \vec{e}^{\,(\varphi)}_b , \qquad \text{(B.3)}$$

$$\vec{K}'_a = V^{-1}_{ab} \vec{e}^{\,(\theta)}_b , \qquad \vec{l}'_a = v^{-1}_{ab} \vec{e}^{\,(\varphi)}_b ,$$

where

$$U^{-1}_{ab}(\theta) = u^{-1}_{ab}(\varphi \to \theta) = -\sin^{-1}\theta_2 \left\| \begin{matrix} \cos\theta_1 \cos\theta_2, & \sin\theta_1 \cos\theta_2, & -\sin\theta_2 \\ \sin\theta_1 \sin\theta_2, & -\cos\theta_1 \sin\theta_2, & 0 \\ -\cos\theta_1, & -\sin\theta_1, & 0 \end{matrix} \right\| , \qquad \text{(B.4)}$$

$$V^{-1}_{ab}(\theta) = v^{-1}_{ab}(\varphi \to \theta) = -\sin^{-1}\varphi_2 \cdot \left\| \begin{matrix} \cos\theta_3, & -\sin\theta_3, & 0 \\ -\sin\theta_2 \sin\theta_3, & -\sin\theta_2 \cos\theta_3, & 0 \\ -\cos\theta_2 \cos\theta_3, & \cos\theta_2 \sin\theta_3, & -\sin\theta_2 \end{matrix} \right\| . \qquad \text{(B.5)}$$

Let us expand rotation vectors over both coordinate systems:

$$\sum_{a=1}^{3} \vec{e}^{\,(\theta)}_a d\theta_a = \sum_{a=1}^{3} \vec{K}_a d\beta_a = \sum_{a=1}^{3} \vec{K}'_a d\alpha_a , \qquad \text{(B.6)}$$

$$\sum_{a=1}^{3} \vec{e}^{\,(\varphi)}_a d\varphi_a = \sum_{a=1}^{3} \vec{l}_a d\delta_a = \sum_{a=1}^{3} \vec{l}'_a \delta\gamma_a$$

and define components of operators of the angular momentum with respect to axes of "moving" and "fixed" systems as follows:

$$\hat{M}_a = -i\frac{\partial}{\partial \alpha_a} = -i\frac{\partial \theta_b}{\partial \alpha_a}\frac{\partial}{\partial \theta_b} = -i V^{-1}_{ba}\frac{\partial}{\partial \theta_b} \quad ,$$

$$\hat{N}_a = -i\frac{\partial}{\partial \gamma_a} = -i\frac{\partial \varphi_b}{\partial \gamma_a}\frac{\partial}{\partial \theta_b} = -i \mathcal{V}^{-1}_{ba}\frac{\partial}{\partial \varphi_b} \quad , \tag{B.7}$$

$$\hat{m}_a = -i\frac{\partial}{\partial \beta_a} = -i\frac{\partial \theta_b}{\partial \beta_a}\frac{\partial}{\partial \theta_b} = -i U^{-1}_{ba}\frac{\partial}{\partial \theta_b} \quad ,$$

$$\hat{n}_a = -i\frac{\partial}{\partial \delta_a} = -i\frac{\partial \varphi_b}{\partial \delta_a}\frac{\partial}{\partial \varphi_b} = -i \mathcal{V}^{-1}_{ba}\frac{\partial}{\partial \varphi_b} \quad .$$

Here we have made use of (B.3) and (B.6). Substituting the explicit form of the matrices (B.4) and (B.5) in (B.7) we obtain the formulae (4.24) and (4.26).

Appendix C

As is known matrices $d^{s}_{s_3 s_3'}(\theta_2)$ have been calculated by Wigner [24] and are equal to

$$d^{s}_{s_3 s_3'}(\theta_2) = \left[(s+s_3)!(s-s_3)!(s+s_3')(s-s_3')!\right]^{1/2} \cdot \sum_{\nu} \frac{(-1)^{\nu}}{(s-s_3-\nu)!(s+s_3'-\nu)!(\nu+s_3-s_3')!\nu!}\left(\cos\frac{\theta_2}{2}\right)^{2s+s_3'-s_3-2\nu} \cdot \left(-\sin\frac{\theta_2}{2}\right)^{s_3-s_3'+2\nu} \tag{C.1}$$

The subscript ν takes all the integral values for which the arguments under the sign of factorial are $\geq$ 0.

Matrix elements of the operators $\hat{m}_a, \hat{n}_a, \hat{M}_a, \hat{N}_a$ for $a=1,2$ in the basis (5.2) are:

$$\langle s\, s_3 \pm 1\, s_3' | \hat{m}_1 | s\, s_3 s_3' \rangle = 1/2 \cdot \sqrt{s(s+1) - s_3(s_3 \pm 1)} \ ,$$

$$\langle s\, s_3 \pm 1\, s_3' | \hat{m}_2 | s\, s_3\, s_3' \rangle = \mp i/2 \cdot \sqrt{s(s+1) - s_3(s_3 \pm 1)} \ ,$$

$$\langle s\, s_3\, s_3' \pm 1 | \hat{M}_1 | s\, s_3 s_3' \rangle = 1/2 \cdot \sqrt{s(s+1) - s_3'(s_3' \pm 1)} \ , \tag{C.2}$$

$$\langle s\, s_3 s_3' \pm 1 | \hat{M}_2 | s\, s_3 s_3' \rangle = \pm i/2 \sqrt{s(s+1) - s_3'(s_3' \pm 1)} \ ,$$

similarly for $\hat{n}_a$ and $\hat{N}_a$. By means of these matrix elements one can easily calculate matrix elements of the operator $\hat{T}$ (5.9):

$$\hat{T}^{s s_3 s_3'}_{s s_3 s_3'} = \frac{1}{4}\left[\frac{I_1}{I_1^2 - J_1^2} + \frac{I_2}{I_2^2 - J_2^2}\right]\left[s(s+1) - s_3'^2\right] + \frac{I_3 s_3'^2}{2(I_3^2 - J_3^2)} \ ,$$

$$\hat{T}^{s s_3 s_3'+2}_{s s_3 s_3'} = \hat{T}^{s s_3 s_3'}_{s s_3 s_3'+2} = \tag{C.3}$$

$$\frac{1}{8}\left[\frac{I_1}{I_1^2 - J_1^2} - \frac{I_2}{I_2^2 - J_2^2}\right] \cdot \sqrt{(s - s_3')(s - s_3' - 1)(s + s_3' + 1)(s + s_3' + 2)} \ .$$

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