ЕФИ-824(51)-85

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒՏ

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

Ts.A. AMATUNI, Kh.N. SANOSSYAN

OPTIMUM SEGMENTATION OF A TARGET FOR NUCLEAR INTERACTION LENGTH MEASUREMENTS

ЦНИИатоминформ

ЕРЕВАН-1985

ЕФИ-824(51)-85

Ц.А.АМАТУНИ, Х.Н.САНОСЯН

ОПТИМАЛЬНАЯ СЕГМЕНТАЦИЯ МИШЕНИ ДЛЯ ИЗМЕРЕНИЯ ПРОБЕГА НЕУПРУГОГО ВЗАИМОДЕЙСТВИЯ АДРОНОВ

Исследована статистическая точность оценки пробега неупругого взаимодействия адронов высокой энергии по распределению точек взаимодействия в сложной мишени, состоящей из чередующихся рядов поглотителя и детекторов. Показана возможность оптимального выбора толщины слоя мишени, при заданном их числе, позволяющая максимально повысить статистическую точность измерений.

Ереванский физический институт
Ереван 1985

ЕФИ-824(51)-85

Ts.A. AMATUNI, Kh.N. SANOSSYAN

OPTIMUM SEGMENTATION OF A TARGET FOR NUCLEAR INTERACTION LENGTH MEASUREMENTS

The statistical accuracy of nuclear interaction length estimation from the distribution of interaction points inside a segmented target, composed of alternating layers of absorbers and detectors, is studied. It is shown that for a given number of target segments the thickness of the absorber layers can be optimized allowing to achieve almost limiting statistical accuracy of measurements.

Yerevan Physics Institute
Yerevan 1985

Introduction

The interaction length of high-energy hadrons in dense media can be measured with a segmented target composed of alternating layers of absorbers and detectors. The parameter that is measured is the segment number of the target in which the first inelastic interaction took place. This is facilitated by the larger amount of energy released in the adjacent detector layer corresponding to the passage of more than one particles. The interaction length can be estimated from a maximum likelihood fit to the observed numbers of particles which have interacted in the different segments [1].

In the present article we study the statistical accuracy of such measurements. In §1 we consider the idealized case of an infinite,"continious" target which allows by assumption to measure exactly the depth of the first interaction inside the target. This simplification allows one to determine the limiting accuracy of the measurements. In §2 the case of the segmented target is studied and it is shown, that for a given number of segments the thickness of the absorber segments can be optimized so as to yield approximately the same accuracy of measurements as that obtained with the "continious" target.

§1. Measurements with an infinite "continious" target

Let us consider first the simple case of an infinite "continious" target which can be regarded as the limiting case of a very long segmented target with very thin layers. In this case the depth of the first inelastic interaction inside the target can be determined exactly. The interaction points are distributed exponentially

$$f(x) = \frac{1}{\lambda_0} \exp(-x/\lambda_0) , \tag{1}$$

where x is the depth of the first interaction and λ_0 is the nuclear interaction length to be estimated. The maximum likelihood (ML) estimate of λ_0 is

$$\hat{\lambda} = \frac{1}{N} \sum_{i=1}^{N} x_i , \tag{2}$$

where $x_i \ (i=1,2,\dots,N)$ are the measured values of x .

Let $F_N(\hat{\lambda} \mid \lambda_0)$ be the probability distribution function of $\hat{\lambda}$ for given λ_0 and N :

$$F_N(\hat{\lambda} \mid \lambda_0) = \int_0^{\infty} \dots \int_0^{\infty} dx_1 \dots dx_N \frac{1}{\lambda_0} e^{-\frac{x_1}{\lambda_0}} \dots \frac{1}{\lambda_0} e^{-\frac{x_N}{\lambda_0}} \delta\Big(\hat{\lambda} - \frac{1}{N} \sum_{i=1}^{N} x_i\Big) . \tag{3}$$

Substituting the Fourier transform of the δ - function

$$\delta\Big(\hat{\lambda} - \frac{1}{N} \sum_{i=1}^{N} x_i\Big) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\Big[iq\Big(\hat{\lambda} - \frac{1}{N} \sum_{i=1}^{N} x_i\Big)\Big] dq$$

into (3) one can find

$$F_N(\hat{\lambda}|\lambda_0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(iq\hat{\lambda})[1+iq\lambda_0/N]^{-N}dq, \tag{4}$$

where the expression in the square brackets is the characteristic function of X/N . The integral in (4) can be evaluated easily yielding

$$F_N(\hat{\lambda}|\lambda_0)d\hat{\lambda}=\frac{1}{(N-1)!}\left(N\frac{\hat{\lambda}}{\lambda_0}\right)^{N-1}\exp\left(-N\frac{\hat{\lambda}}{\lambda_0}\right)d\left(N\frac{\hat{\lambda}}{\lambda_0}\right) \tag{5}$$

The expectation value, the variance and the assymetry and excess parameters of $\hat{\lambda}$ are:

$$E(\hat{\lambda})=\int_0^{\infty}\hat{\lambda}F_N(\hat{\lambda}|\lambda_0)d\hat{\lambda}=\lambda_0, \tag{6a}$$

$$\mathcal{D}(\hat{\lambda})=\int_0^{\infty}(\hat{\lambda}-\lambda_0)^2F_N(\hat{\lambda}|\lambda_0)d\hat{\lambda}=\lambda_0^2/N=1/I_c, \tag{6b}$$

$$\gamma_1=2/\sqrt{N}, \quad \gamma_2=6/N, \tag{6c}$$

where I_c is Fisher's information. It follows from eg. (5-6) that the maximum likelihood estimate (2) is consistent, unbiased and efficient [2] , and as a consequence of the latter the relative error,

$$\sigma(\hat{\lambda})/\lambda_0=1/\sqrt{N}, \tag{7}$$

where $\sigma(\hat{\lambda})=\sqrt{\mathcal{D}(\hat{\lambda})}$ denotes the r.m.s deviation, is the maximum achiveable accuracy for N independent measurements. We shall see in the sequal that due to the finite size of the target and the sampling rate the statisyical accuracy deteriorates significantly unless an optimum choice of segment thickness is made.

§2 Measurements with a segmented target

Let us consider now the case when the target consists of K layers of absorbers each ΔX thick, the total thickness of the target being $X_0 = K \cdot \Delta X$ (we neglect the thickness of detectors, which are interleaved between absorber layers). The probability, that the particle will interact inelastically in the i-th segment of the target after passing (i-1) segments without an interaction, normalized to the total probability of interacting inside the target, is equal to

$$P_i(\lambda_0) = \exp(-x_i/\lambda_0)\,\frac{1-\exp(-\Delta x/\lambda_0)}{1-\exp(-x_0/\lambda_0)} \tag{8}$$

where $x_i = (i-1)\Delta x, i = 1,2,\ldots, K$ *). The likelihood function has the form

$$L(\lambda) = N!\prod_{i=1}^{K} P_i^{N_i}(\lambda)/N_i!\ , \tag{9}$$

where N_i is the experimentally measured number of interactions in the i-th segment of the target, $N = \sum_{i=1}^{K} N_i$ is the total number of interactions inside the target. For a fixed N, $\{N_1, \ldots, N_K\}$ are variates from a polynomial distribution.

---

*) In the first approximation the angular spread of the beam particles can be accounted for by substituting $\Delta x \rightarrow \Delta x/\langle\sec\theta\rangle$ into the subsequent formulas (θ is the incidence angle).

Substituting the expression for P_i from (8) into (9) one can find

$$L(\lambda)=\frac{N!}{N_1!\,N_2!\cdots N_K!}\exp(-S/\lambda)\left[\frac{1-\exp(-\Delta x/\lambda)}{1-\exp(-x_0/\lambda)}\right]^N \tag{10}$$

where the statistics

$$S=\sum_{i=1}^{K} N_i x_i \tag{11}$$

is introduced.

The maximum likelihood equation will read

$$\frac{\partial \ln L(\lambda)}{\partial \lambda}\equiv\frac{1}{\lambda^2}\left[S-\varphi(\lambda)\right]=0\ , \tag{12}$$

where

$$\varphi(\lambda)=N\left(\frac{\Delta x}{e^{\Delta x/\lambda}-1}-\frac{x_0}{e^{x_0/\lambda}-1}\right) \tag{13}$$

is the expectation value of S

$$E(S)=\sum_{i=1}^{K}E(N_i)x_i=N\sum_{i=1}^{K}P_i(\lambda)x_i=\varphi(\lambda)\,. \tag{14}$$

From (10) and (12) it follows [3], that S is a sufficient statistics for λ and an unbiased sufficient estimate of $\varphi(\lambda)$. For $K=2$ and $K=\infty$ equation (12) can be solved analytically, otherwise - only numerically.

For a given N the statistics S has a discrete set of possible values:

$$S=S_j=j\Delta x\ ,\quad j=0,1,\ldots,N(K-1)\,. \tag{15}$$

For some values of S from (15) the likelihood function has no extremum and achieves it's maximum at $\lambda=\infty$. Indeed.

$\varphi(\lambda)$ increases monotonically from 0 for $\lambda=0$ to $\frac{1}{2}S_{max}= = \frac{1}{2}N(K-1)\Delta X$ at $\lambda=\infty$, hence, equation (12) has a solution (and if so–a single one), only for $S<\frac{1}{2}S_{max}$. When $S\geqslant\frac{1}{2}S_{max}$, which, as it is easy to see, corresponds to the sequence $N_1\leqslant N_2\leqslant\cdots\leqslant N_K$, equation (12) has no finite solution, but the probability of such an outcome is neglegibly small for $N\gg 1$. The estimated value of λ is a monotonically increasing function of S (see eq. (12–13)), and hence it also has a discrete distribution. For given N and K, $\hat{\lambda}$ can take on $\nu=\left[\frac{1}{2}(N(K-1)+1)\right]$ discrete values:

$$\hat{\lambda}=\hat{\lambda}_j=\varphi^{-1}(S_j)\ ;\quad S_j=j\Delta X\ ;\quad j=0,1,2,\dots,\nu-1\ , \tag{16}$$

where φ^{-1} is the inverse function with respect to φ .

The sampling distribution of $\hat{\lambda}$ (sample size–10^5) for $N=20$, $\Delta X=80\,gcm^{-2}$, $K=7$ and $\lambda_o=130\,gcm^{-2}$ obtained by Monte-Carlo simulation is given in fig. 1a. The intervals between the adjacent values of $\hat{\lambda}$ decrease with increasing N . The values of $\hat{\lambda}$ obtained by Monte-Carlo simulation for $N=500$ are histogramed in fig.1b where some traces of the discreteness of the true distribution still be seen.

The power series expansion in $1/N$ of the ML-estimate variance $\mathcal{D}(\hat{\lambda})$ and bias $\mathcal{B}(\hat{\lambda})=E(\hat{\lambda}-\lambda_o)$ for the general case when the likelihood function has the form (9) has been obtained in ref. [4]

$$\mathcal{D}(\hat{\lambda})=\frac{1}{NA_1}+O(N^{-2}) \tag{17}$$

$$\mathcal{B}(\hat{\lambda})=-\frac{B_1}{2NA_1^2}+O(N^{-2}) \tag{18}$$

where

$$A_1=\sum_{i=1}^{K}\frac{1}{P_i}\left(\frac{\partial P_i}{\partial\lambda_0}\right)^2 \; {}^{*)} , \quad B_1=\sum_{i=1}^{K}\frac{1}{P_i}\frac{\partial P_i}{\partial\lambda_0}\cdot\frac{\partial^2 P_i}{\partial\lambda_0^2} \; . \qquad (19)$$

Substituting the expression for P_i from (8) into (17-19) and neglecting the terms $O(N^{-2})$ one can find

$$\sqrt{N}\,\sigma(\hat\lambda)/\lambda_0=\left[\left(\frac{\Delta z}{sh\Delta z}\right)^2-\left(\frac{z_0}{sh\,z_0}\right)^2\right]^{-1/2} , \qquad (20)$$

$$N b(\hat\lambda)/\lambda_0=\frac{\left(\frac{\Delta z}{sh\Delta z}\right)^2(1-\Delta z\,cth\Delta z)-\left(\frac{z_0}{sh\,z_0}\right)^2(1-z_0 cth\,z_0)}{\left[\left(\frac{\Delta z}{sh\Delta z}\right)^2-\left(\frac{z_0}{sh\,z_0}\right)^2\right]^2} \qquad (21)$$

where the dimensionless parameters $\Delta z=\Delta X/2\lambda_0$, $z_0=X_0/2\lambda_0$ are introduced.

The dependence of $\sqrt{N}\,\sigma(\hat\lambda)/\lambda_0$ and $N b(\hat\lambda)/\lambda_0$ on Δz (i.e. on ΔX measured in $2\lambda_0$ units) for different values of K are presented on fig.s 2 and 3. One can see that:

1) $\sqrt{N}\,\sigma(\hat\lambda)/\lambda_0>1$ for all K and Δz, and $b(\hat\lambda)\neq 0$ in contrast with the "continious" case (see §1). The ML-estimate in unbiased and efficient only asymptotically,

2) for a given K, $\sqrt{N}\,\sigma(\hat\lambda)/\lambda_0$ and $N|b(\hat\lambda)|/\lambda_0$ have rather wide minima, i.e. an optimum choice of segment thickness $\Delta X=\Delta X^*(K)$ is possible for which the variance and the bias of the estimate

---

*) $NA_1\equiv E[(\partial \ln L/\partial\lambda_0)^2]$ is Fisher's information.

are minimal. The values of $\Delta X^*(K)$ (in λ_0 units) for which $\sqrt{N}\,\sigma(\hat{\lambda})/\lambda_0$ achieves its minimum are presented in the table below, along with the corresponding values of $X_0 = K\cdot\Delta X^*(K)$,

Table

K	$\Delta X^*(K)$	$X_0^*(K)$	$\sqrt{N}\,\sigma(\hat{\lambda})/\lambda_0$	$N\lvert\beta(\hat{\lambda})\rvert/\lambda_0$
2	2.40	4.80	1.51	$1.1\cdot10^{-2}$
3	1.88	5.64	1.253	$6.6\cdot10^{-4}$
4	1.56	6.24	1.160	$4.8\cdot10^{-3}$
5	1.34	6.70	1.113	$8.0\cdot10^{-3}$
6	1.20	7.20	1.085	$4.3\cdot10^{-3}$
7	1.08	7.56	1.067	$4.7\cdot10^{-3}$
8	0.98	7.84	1.054	$2.4\cdot10^{-3}$
9	0.90	8.10	1.045	$1.7\cdot10^{-3}$
10	0.82	8.20	1.038	$5.7\cdot10^{-3}$
100	0.14	14.00	1.001	$6.5\cdot10^{-4}$
∞	0	∞	1.000	0

$\sqrt{N}\,\sigma(\hat{\lambda})/\lambda_0$ and $N\lvert\beta(\hat{\lambda})\rvert/\lambda_0$. From the values given there one can see, that the statistical accuracy of measurements with an optimal choice of ΔX is almost the same as that for the ideal case (the last row in the table). For an optimum choice of ΔX the value of λ_0 must be known. If there is no apriori information on the magnitude of λ_0 (theoretical and/or experimental) then preliminary measurements with an arbitrarily chosen ΔX must be performed first.

The relationships (20-21) are correct only asymptotically for $N \gg 1$. Monte-Carlo simulations show that the asymptotical values are achieved for $N \geqslant (1 \div 10)/P_K$. This is illustrated in fig.4 where the simulation results for $K=7$, $\Delta X = 80 gcm^{-2}$ and $\lambda_0 = 130 gcm^{-2}$ are presented.

The expressions for $\mathcal{D}(\hat{\lambda})$ and $b(\hat{\lambda})$ (20-21) can be used in practice for calculating the variance and the bias of the ML-estimate. For that end, naturally, it is necessary to substitute the estimated value $\hat{\lambda}$ instead of λ_0 into (20-21).

Acknowledgements

We are indepted to E.A.Mamidjanyan from conversations with whom the conception of these notes rised, to V.V.Avakyan for stimulating discussions and to P.Yu. Dallakyan for valuable comments.

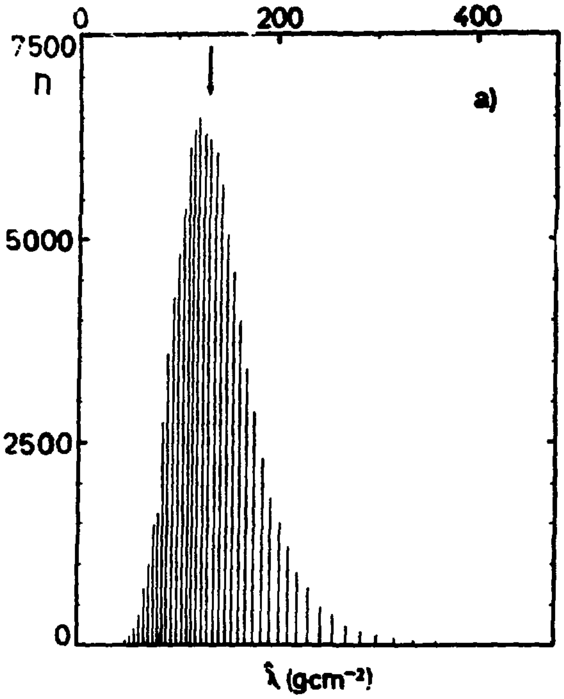
0
200
400
7500
n
a)
5000
2500
0
λ (gcm⁻²)

Fig.1a

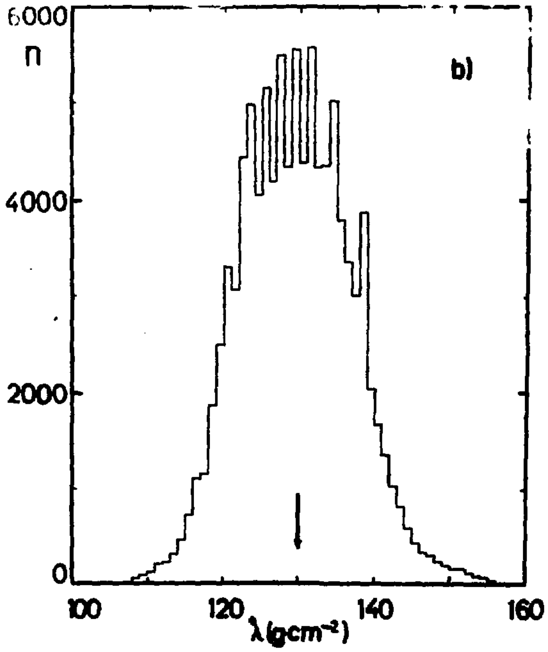
6000
n
b)
4000
2000
0
100
120
140
160
λ(gcm⁻²)

Fig.1b

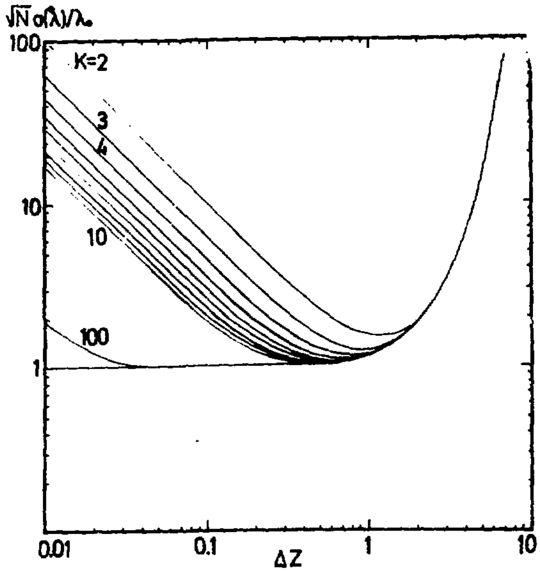
√N σ(λ̂)/λ₀
100
10
1
K=2
3
4
10
100
0.01
0.1
1
10
ΔZ

Fig.2

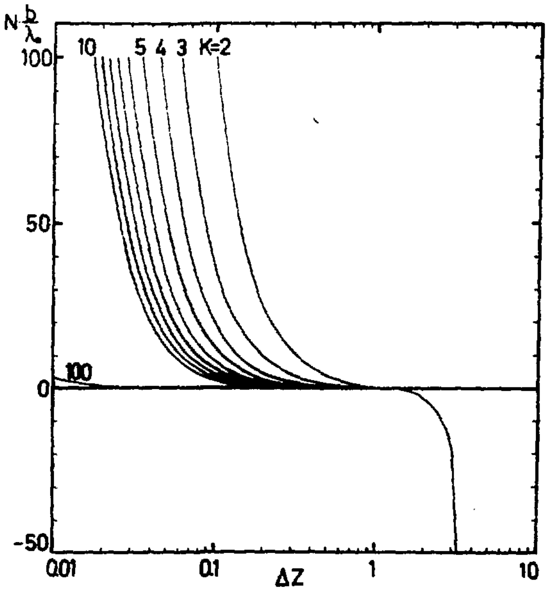
N b/λ₀
100
50
0
-50
10
5
4
3
K=2
100
0.01
0.1
1
10
ΔZ

Fig.3

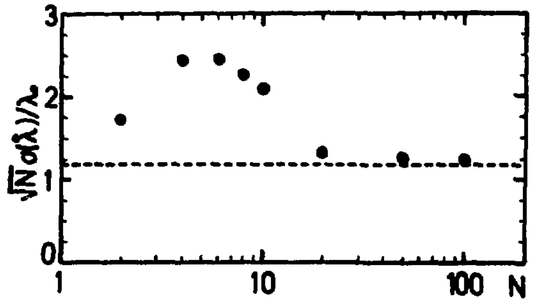

Fig.4

Figure Captions

Fig.1. The sampling distribution of the ML-estimate obtained by Monte-Carlo simulation for $K=7, \Delta X = 80\,gcm^{-2}$ and $\lambda_0 = 130\,gcm^{-2}$ (denoted by an arrow). The sample size is 10^5 for fixed N; a) N=20, b) N=500.

Fig.2. The dependence of the reduced relative error $\sqrt{N}\sigma(\hat{\lambda})/\lambda_0$ on $\Delta Z = \Delta X/2\lambda_0$. The numbers near the curves denote the number of target segments.

Fig.3. The dependence of the reduced relative bias $N b(\hat{\lambda})/\lambda_0$ on $\Delta Z = \Delta X/2\lambda_0$. The numbers near the curves denote the number of target segments.

Fig.4. The dependence of the reduced relative error $\sqrt{N}\sigma(\hat{\lambda})/\lambda_0$ on the total number N of interactions inside the target obtained by Monte-Carlo simulation for $K=7$, $\Delta X = 80\,gcm^{-2}$ and $\lambda_0 = 130\,gcm^{-2}$. The dotted line is the asymptotical value (see formula (20) of §2).

References

1. Janossy L. Theory and practice of the evaluation of measurements. Oxford Univ. press, 1965.
2. Eadie W.T., Dryard D., James F.E., Roos M. Statistical methods in experimental physics, North-Holland, 1971.
3. Kendall M.G., Stuart A. The advanced theory of statistics. Inference and Relationship, Charles Griffin and Co.Ltd., London, vol.2, 1967.
4. Haldane B.S., Smith S.M. The sampling distribution of a maximum likelihood estimate. Biometrika, 1956, vol.43, p.96.

The manuscript was received 31 May 1985

Ц.А.АМАТУНИ, Х.Н.САНОСЯН

ОПТИМАЛЬНАЯ СЕГМЕНТАЦИЯ МИШЕНИ ДЛЯ ИЗМЕРЕНИЯ
ПРОБЕГА НЕУПРУГОГО ВЗАИМОДЕЙСТВИЯ АДРОНОВ

(на английском языке, перевод авторов)

Редактор Л.П.Мукаян
Технический редактор А.С.Абрамян

Подписано в печать 29/УШ-85г. ВФ-06799 Формат 60х84/I6
Офсетная печать. Уч.изд.л. I,0 Тираж 299 экз. Ц. I5 к.
Зак.тип.№ 396 Индекс 3624

Отпечатано в Ереванском физическом институте
Ереван 36, Маркаряна 2

индекс 3624

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ