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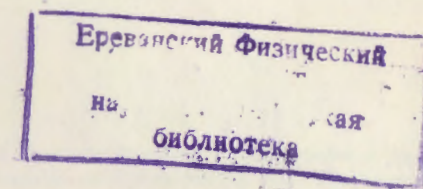
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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TYPICAL PROPERTIES OF DYNAMICAL SYSTEMS  
WITH NON-NEGATIVE CURVATURE



ЦНИИАтоминформ

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## 1. Introduction

The investigation of non-linear dynamical systems during the last decades has led to the discovery of unexpected appearance of stochasticity and regularity. It turns out that both these phenomena are peculiar to almost all physical systems. Considered up to now physical problems include a large class of features - from elementary particles to global properties of the Universe [1,2]. Thus, it was shown that stochasticity is peculiar to non-Abelian Yang-Mills fields [3-5]. The important role of stochasticity is becoming evident in the variety of stellar configurations observed in the Universe - from globular clusters to giant galaxies [6-9].

The most profound properties of dynamical systems are studied in ergodic theory [10-12], where, in particular the classification of systems by the increasing degree of statistical properties is developed. Maximally strong statistical properties possess Anosov U-systems (C-systems) [13], which are characterized by exponential instability of trajectories at every point of space. A number of necessary and in some cases sufficient conditions are obtained defining the degree of stochasticity of the system; e.g. the criterion the system to be U-system is the negativity of the curvature of the

space in all two-dimensional directions.

At the same time, in too many cases there are absent any check-up conditions pointing out what type the given system belongs to. An important class of such systems are those for which the mentioned two-dimensional curvature is non-negative. An example can be the well-known Henon-Heiles two-dimensional system [14], being a model of spiral galaxy; the curvature for it is strongly positive.

In the present paper we investigate the systems with non-negative two-dimensional curvature, including their chaotic and regular properties and the possibility of their reckoning to distinct class of dynamical systems.

It is shown that almost all systems with non-negative curvature are "regular-stochastic" systems (RS-systems), i.e. the existence of regions of stochastic and regular (ordered) motion of positive measure is their typical property. In stochastic regions consisting of several ergodic components with positive entropy, these systems have no transversal fibers and therefore seem not to be K-systems. The existence of strongly stable solutions of the Henon-Heiles system is shown. This fact explains the appearance of the islands of stability on the Poincare sections obtained by numerical methods. The found results point out the rich variety of possible stellar motions in spiral galaxies of different classes. It seems that the ratio of measures of stochastic and ordered regions has a crucial role in determining the class of the spiral galaxy.

The content of the paper is as follows. In Sections 2 and 3 we summarize some of the background ideas and notations from differential geometry and ergodic theory we shall make use of. In Section 4 the Hamiltonian system is reduced to geodesical flow on a Riemannian manifold and the definition of strong stability is given. In Section 5 the properties of integrable and near-integrable systems are discussed. The existence of strongly stable

solutions of Henon-Heiles system is shown in Section 6. In Section 7 the typical properties of dynamical systems are shown up. The obtained results are discussed in Section 8.

## 2. Geodesical Flow and the Jacobi Equation

The detailed discussion of the conceptions of differential geometry collected below one can find in [15-17].

Let  $M$  be a smooth (of class  $C^\infty$ )  $N$ -dimensional compact manifold with a defined Riemannian metric denoted by  $\langle \cdot, \cdot \rangle$ . Denote by  $TM$   $2N$ -dimensional tangent space to the manifold  $M$  and its projection - by  $\pi: TM \rightarrow M$ . The existence of Riemannian metric enables one to introduce different objects and structures in  $M$ . We shall now give their definitions, notations and only some of them discuss at greater length.

The introduced metric defines a norm  $\|\cdot\|$  in each tangent space  $T_x M$  ( $x$  is a point in  $M$ ), a Riemannian connection (Levi-Civita connection) and induces covariant derivative denoted by  $\nabla$ . curvature tensor

$$R(u, n)v = (\nabla_n \nabla_u - \nabla_u \nabla_n + \nabla_{[n, u]})v, \quad (2.1)$$

twisting tensor

$$T(u, n) = \nabla_n u - \nabla_u n + [n, u] \equiv 0,$$

where

$u, v, n$  are smooth vector fields in  $M$  or along a distinct curve.

Geodesics are those curves along which the tangent vector field is parallel. We shall assume the length of the curve  $S$  to be the parameter of the geodesic  $\gamma(s)$ , so we have

$$\|u\|=1, \nabla_u u = 0, \text{ where } u = \frac{d}{ds} \gamma(s).$$

Geodesics are infinitely continuable on a compact manifold, so that can take all values from  $-\infty$  to  $+\infty$ .

For every  $X$  from  $M$  and two-dimensional subspace  $\mathcal{G}$  of  $T_X M$  (given by vectors  $u, v$  on  $T_X M$ ) the value

$$K_{u,v}(x) = \frac{\langle R(u,v)u, v \rangle}{\|u\|^2 \|v\|^2 - \langle u, v \rangle^2} \quad (2.2)$$

is defined called two-dimensional curvature of the manifold  $M$  in the direction  $\mathcal{G}$  at the point  $X$  (one can show that  $K_{u,v}(x)$  does not depend on  $u, v$  defining the surface  $\mathcal{G}$ ). For two-dimensional manifolds the curvature  $K_{u,v}(x) = K(x)$  coincides with Gaussian one. The Riemannian metric is said to have negative (non-negative) curvature if the latter is negative (non-negative) at every point and in every two-dimensional direction.

Consider a family of geodesics smoothly dependent of parameter  $\lambda$  (Fig.1)

$$u = \frac{d}{ds} \Big|_{\lambda = \text{const}}, \quad n = \frac{d}{d\lambda} \Big|_{s = \text{const}},$$

then

$$[u, n] = 0.$$

Differentiating the equation of geodesics  $\nabla_u u = 0$  by  $n$  and using the following expressions

$$R(u, n)v = ([\nabla_n, \nabla_u] + \nabla_{[n, u]})v = [\nabla_n, \nabla_u]v,$$

$$\nabla_n u - \nabla_u n = [u, n] = 0,$$

we have

$$0 = \nabla_n \nabla_u u = \nabla_u \nabla_n u + [\nabla_n, \nabla_u] u$$

or

$$\nabla_u \nabla_u n + R(u, n)u = 0. \quad (2.3)$$

The equation (2.3) defining the deviation of closely geodesics is called Jacobi equation. The vector field  $n(s)$  satisfying the Jacobi equation is called Jacobi field. The Jacobi field is orthogonal to the velocity vector (shortly, simply orthogonal), if  $\langle n(s), u(s) \rangle = 0$  for every  $s$ . Then  $\langle \nabla_u n, u \rangle = 0$  for every  $s$ .

Denote

$$A(s) = \langle u(s), n(s) \rangle,$$

$$\frac{dA(s)}{ds} = A'(s),$$

then we have

$$A'(s) = \frac{d}{ds} \langle u, n \rangle = \nabla_u \langle u, n \rangle = \langle u, \nabla_u n \rangle,$$

$$A''(s) = \langle u, \nabla_u \nabla_u n \rangle = -\langle u, R(u, n)u \rangle = 0,$$

and if

$$A(0) = A'(0) = 0,$$

then

$$A(s) = 0$$

for every  $s$ .

I.e., if

$$\langle u, n \rangle \Big|_{s=0} = \langle u, \nabla_u n \rangle \Big|_{s=0} = 0,$$

one has the orthogonality condition

$$\langle u, n \rangle = 0$$

for every  $s$ .

Below we shall assume that the Jacobi field is orthogonal.

One can derive an equation for  $\|n\|^2$  from the Jacobi equation (2.3).

Multiplying Eq.(2.3) by  $2n$ , we have

$$2 \cdot \langle \nabla_u \nabla_u n, n \rangle + 2 \cdot \langle R(u, n)u, n \rangle = 0,$$

$$\frac{d^2 \|n\|^2}{ds^2} = \nabla_u \nabla_u \|n\|^2 = 2 \cdot \nabla_u \langle n, \nabla_u n \rangle = 2 \|\nabla_u n\|^2 + 2 \langle n, \nabla_u \nabla_u n \rangle$$

or

$$\frac{d^2 \|n\|^2}{ds^2} = -2 \langle R(u, n)u, n \rangle + 2 \|\nabla_u n\|^2. \quad (2.5)$$

So far as Jacobi field is orthogonal, from (2.2) we come to

$$K_{u, n}(\gamma(s)) = K_{u, n}(s) = \frac{\langle R(u, n)u, n \rangle}{\|n\|^2},$$

$$(\|u\|=1, \langle u, n \rangle = 0 \text{ for every } S : (2.4)).$$

Then Eq.(2.5) turns to

$$\frac{d^2 \|n\|^2}{ds^2} = -2 \cdot K_{u, n}(s) \cdot \|n\|^2 + 2 \cdot \|\nabla_u n\|^2$$

or

$$\frac{d^2 \|n\|^2}{ds^2} \geq -2 \cdot K_{u, n}(s) \cdot \|n\|^2. \quad (2.6)$$

The inequality (2.6) rewritten in a form

$$\frac{d^2 \|n\|^2}{ds^2} \geq -2 \cdot k \|n\|^2$$

as a solution

$$\|n(s)\| \geq \frac{1}{2} \|n(0)\| \exp(\sqrt{-2k} s) \text{ at } s > 0, \quad (2.7)$$

where

$$k = \max_{(x, u, n)} (K_{u, n}(x)) < 0.$$

The expression (2.7) shows the divergence of closely geodesics with a rate not slower than the exponential one. As it was proved by Anosov [13], these systems with negative curvature are U-systems (for details, see below).

Further we shall study two-dimensional manifolds.

Let us show that in this case the Jacobi equation has a form

$$z''(s) + K(s)z(s) = 0, \quad (2.8)$$

where  $n(s) = z(s) \cdot \hat{n}(s)$ ,  $\|\hat{n}\|=1$ ,  $K(s)$  is the Gaussian curvature of on the point  $S$  on geodesic  $\gamma(s)$ . So far as Jacobi field is orthogonal, then

$$\langle n, u \rangle = \langle \nabla_u n, u \rangle = 0 \text{ for every } S \text{ (2.4) or}$$

$$\langle \hat{n}, u \rangle = \langle \nabla_u \hat{n}, u \rangle = 0,$$

so far as

$$\nabla_u n = z' \cdot \hat{n} + z \cdot \nabla_u \hat{n}.$$

From

$$\|\hat{n}\|=1$$

and

$$\langle \nabla_u \hat{n}, \hat{n} \rangle = 0,$$

we have that

$$\nabla_u \hat{n} \perp \hat{n}, \nabla_u \hat{n} \perp u, \hat{n} \perp u.$$

So far as in two-space three non-zero vectors  $u, \hat{n}, \nabla_u \hat{n}$  cannot be mutually orthogonal. we have

$$\nabla_u \hat{n} = 0, \quad (\|u\| = \|n\| = 1).$$

Using the relations

$$\nabla_u u = \nabla_u (z \hat{n}) = z' \hat{n},$$

$$\nabla_u \nabla_u n = z'' \hat{n},$$

$$R(u, n)u = z \cdot R(u, \hat{n})u,$$

from the Jacobi equation (2.3) we come to

$$z'' \hat{n} + R(u, \hat{n})u \cdot z = 0$$

or

$$z'' + \langle R(u, \hat{n})u, \hat{n} \rangle \cdot z = 0,$$

from which the equation (2.8) just follows.

Every point  $\xi$  on  $TM$  is defined by a pair  $(x, u)$  of a point from  $M$  and tangent vector  $u$  from  $T_x M$ . The point  $\xi$  is from  $S_x M$  if  $\|u\| = 1$ .

Let  $\gamma$  be a geodesics having on a point  $X(0)$  a tangent vector  $u(0)$  from  $S_{X(0)} M$ . Let the point  $X(s)$  move in positive direction with unit velocity  $\|u(s)\| = 1$ . At every moment  $s$  a velocity vector  $u(s)$  in  $S_{X(s)} M$  is defined. The transformation  $f^s: SM \rightarrow SM$  is a geodesical flow in  $SM$  if

$$f^s(x(0), u(0)) = (x(s), u(s)).$$

### 3. Classification of Dynamical Systems

The smooth ergodic theory (see [10, 12]) inquires into the ergodic properties of smooth dynamical systems on smooth compact Riemannian manifolds

preserving given normalized measure  $\mu$  compatible with smoothness (i.e. equivalent to Riemannian volume). Under dynamical system of class  $C^r$  on manifold  $M$  we understand differentiable by  $t$  (from  $R$ ) one-parametric group  $f^t$  of diffeomorphisms of  $M$  (dynamical system with continuous time or flow), defined by vector field  $X$  of class  $C^r$  on  $M$  so that

$$\frac{df^t(x)}{dt} = X(x), \quad (X \text{ on } M).$$

An important achievement of ergodic theory is the classification of flows by statistical properties: those are ergodic systems, systems with mixing, K-systems, Bernoulli systems.

The dynamical system is called ergodic one if for any invariant subset

$A$  its measure  $\mu(A)$  is 0 or 1, i.e. if

$$f^t A = A = f^{-t} A,$$

so that

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1. \quad (3.1)$$

For ergodic systems for almost every point  $X$  from  $M$  the average by time equals the average by phase space. Hence, almost all trajectories of ergodic system pass close enough from almost any points of  $M$ .

The systems with mixing possess the following property: any subset of  $M$  varies in accord to equations of motion in such a way preserving its measure and connection, that the measure of its part being in every subset aspires with time to be proportional to the measure of the mutually fixed subset, i.e. tends to an equilibrium state. Therefore for systems with mixing one can speak on relaxation process. Ergodic systems do not possess this property: for them asymptotic values are reached on the average by time only.

Sufficiently stronger statistical properties are possessed by Kolmogorov

K-systems [18], which are characterized by tending to equilibrium state by exponential rate (systems with mixing tend to equilibrium by the power law). For K-systems one can introduce relaxation time defined by the exponential index. It is easy to show that a system is a K-system iff it does not admit any determined measuring besides having probability 0 or 1 (Kolmogorov's 0-1 law). K-systems possess the properties of mixing, ergodicity and have positive KS-entropy (after Krylov [19], Kolmogorov [18] and Sinai [20]). KS-entropy can be defined as follows (see [1]): divide at moment  $t=0$  the space into  $\{A_j(0)\}$  cells of finite measure and consider their evolution back in time within unit time interval. The result will be a new  $\{A_j(-1)\}$ . Every element of intersection of these two sets  $B(-1) = \{A_j(0) \cap A_j(-1)\}$  as a rule has a smaller measure than the element  $A_j(0)$ . Continuing this process so that the measure of element  $B(-t)$  decreases exponentially when  $t \rightarrow +\infty$ , one can show that the following condition will be satisfied

$$h_A(\{A_j(0)\}) = \lim_{t \rightarrow +\infty} \frac{1}{t} \sum_j \mu[B_j(-t)] \ln[B_j(-t)] > 0, \quad (3.2)$$

the KS-entropy  $h$  is the upper limit  $h_A$  for every initial partitions of the phase space. Well-known examples of K-systems are systems of the billiard type [21].

Bernoulli systems are characterized by property of passing the trajectory through different cells of phase space to be a statistically independent process, i.e. it does not admit any deterministic measurings.

Strong statistical properties possess the above-mentioned U-systems, introduced and studied by Anosov [13]. These systems will be discussed in detail in Section 8. Here we only note that geodesical flow on a closed compact Riemannian manifold of negative curvature is an U-system. This flow is isomorphic to Bernoulli flow [12, 22]; in particular it is ergodic.

possesses the property of mixing of all degrees and the K-property.

It is not always possible to obtain the class the given system belongs to. But there exists a property which is possible to check. First, we give some more definitions.

Lyapunov characteristic number for flow  $f^S$  is called a function  $\chi: TM \rightarrow \mathbb{R}$  defined by equality (see [23])

$$\chi(x, v) = \lim_{s \rightarrow +\infty} \frac{1}{s} \ln \|df_x^s(v)\|, \quad (3.3)$$

( $x$  in  $M$ ,  $v$  in  $T_x M$ ), where the existence of unique limit is guaranteed by Oseledec's theorem [24].

The flow  $f^S$  is called having non-zero Lyapunov numbers if the set

$$\Lambda = \{x \text{ in } M: \chi(x, v) \neq 0 \text{ for some } v \text{ from } T_x M\}$$

is positive measured:  $\mu(\Lambda) > 0$ .

One can show that for every  $X$  the supremum of functions  $\chi$  on subspace  $T_x M$  takes no more than  $N$  values different from  $-\infty$ . Denote it by increasing order:

$$\chi_1(x) < \chi_2(x) < \dots < \chi_{a(x)}(x) \quad a(x) \leq N.$$

Then from Pesin formula [23]

$$h(f) = \int_M \sum_{\chi_i > 0} \chi_i(x) d\mu(x), \quad (3.4)$$

it follows that a flow with non-zero Lyapunov characteristic numbers has positive KS-entropy.

Trajectory of a system is called regular (ordered) if all  $\chi_i = 0$ , and it is called stochastic if  $\chi_i \neq 0$  exist on that trajectory. A system is called stochastic if  $h > 0$ , or, the same,  $\mu(\Lambda) > 0$ . The system having regular and stochastic trajectories of positive measure we shall call

RS-systems. Below we show that in some sense almost every system with is an RS-system.

#### 4. Reduction of Hamiltonian System to a Geodesical Flow and the Property of Strong Stability

Let  $T^*M$  be a  $2N$ -dimensional cotangent fiber to the manifold  $M$ , i.e. a space of differential 1-forms in  $M$ . Let  $U$  be a coordinate subset with coordinates denoted by  $q^1, \dots, q^N$ . Then every 1-form on  $U$  is defined by  $N$  components  $p_1, \dots, p_N$ . Non-degenerate differential 2-form  $\omega = dp \wedge dq$  fixes a symplectic structure on  $T^*M$ . For every smooth function  $H(p, q)$  on  $T^*M$  one can construct a system of differential equations which for variables  $p, q$  has a form

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad j = 1, \dots, N. \quad (4.1)$$

Function  $H$  is called a Hamiltonian, the system of equations (4.1) - Hamiltonian system. Choose  $H$  in a form

$$H(p, q) = \sum_{j=1}^N \frac{p_j^2}{2} + V(q^1, \dots, q^N),$$

so far as  $H$  does not depend on time, then

$$\frac{dH(p, q)}{dt} = 0,$$

i.e.  $H(p, q) = E$  is an integral of motion.

The solution of Eq.(4.1)

$$(p(t), q(t)) = f^t(p(0), q(0))$$

defines a flow  $f^t$  in phase space  $T^*M$ .

To investigate the stochastic properties of the flow  $f^t$  in phase space  $T^*M$ , besides Eq.(4.1) one should proceed from equations in variations:

$$\frac{d(\delta q^i)}{dt} = \frac{\partial^2 H}{\partial q^m \partial q^i} \delta q^m + \frac{\partial^2 H}{\partial p_m \partial p_i} \delta p_m,$$

$$i = 1, \dots, N \quad (4.2)$$

$$\frac{d(\delta p_i)}{dt} = -\frac{\partial^2 H}{\partial q^m \partial q^i} \delta q^m - \frac{\partial^2 H}{\partial p_m \partial p_i} \delta p_m.$$

Denote  $\delta = \left[ \sum_i (\delta q^i)^2 + (\delta p_i)^2 \right]^{1/2}$ ; then the Lyapunov characteristic number will be

$$\chi(p, q, \delta p, \delta q) = \lim_{t \rightarrow +\infty} \frac{\ln |\delta|}{t}.$$

It is easy to see that for Hamiltonian systems at least two Lyapunov numbers are equal to zero (one of them corresponds to the vector along the trajectory, the other is zero due to the fact that on energetic  $N-1$ -dimensional hypersurface a shift perpendicular to it is absent). Therefore it is worth to proceed from equations, where these "inconveniences" are absent. Later on it will become clear that the derived new equations are more convenient and effective.

By means of Maupertuis principle the phase flow of Hamiltonian system may be represented as a geodesical flow in a configurational space  $M$  with a Riemannian metric [23]

$$ds^2 = [E - V(q^1, \dots, q^N)] \sum_{j=1}^N (dq^j)^2, \quad (4.3)$$

where the configurational space is

$$M = \{ q, E - V(q) > 0 \}.$$

The equations of the system would be equations of geodesics on  $M$  with a metric (4.3):

$$\nabla_u u = 0 \quad (u \text{ from SM}). \quad (4.4)$$

The affine parameter on the geodesic is related with true (physical) time by equation

$$ds = \sqrt{2} (E - V(\varphi)) dt. \quad (4.5)$$

Then Eq.(4.4) coincides with Eq.(4.1), and the condition  $\|u\| = 1$  ( $u$  from SM) - with that of  $H(p, q) = E$ .

According to Liouville theorem the invariant measure for geodesical flow can be defined as follows:

$$d\mu = d\sigma d\vartheta,$$

where  $d\vartheta$  is the measure on M,  $d\sigma$  is an element of unit (N-1)-dimensional sphere.

One can see that if  $C(\cdot)$  is the phase trajectory of the system with  $C(0) = (p(0), q(0))$ , then the geodesic will be the curve

$$\gamma(\cdot) = \pi C(\cdot). \quad (4.6)$$

The trajectories on M could be represented as follows: material point moves on M with a velocity  $\|u\| = 1$  (i.e. by geodesic). Its motion is defined by the metric until the point reaches the boundary  $\partial M$  of manifold M. Although the metric has singularity at  $\partial M$ ;  $ds|_{\partial M} = 0$ , from (4.6) one can see that the trajectory of the point is completely defined.

Let us show that one can approach the given point on  $\partial M$  by unique trajectory only. Indeed, so far as on  $\partial M$ ,  $E = V(\varphi)$  and  $p_i = 0$  ( $i = 1, \dots, N$ ), i.e. one can pass by the point, say,  $q_0^i$  on  $\partial M$ , if passing through the point  $(0, q_0)$  in the phase space; however only one trajectory passes through any given point of the phase space.

Therefore, although the metric has a singularity on  $\partial M$ , the geodesics

are uniquely defined. Later on we shall consider M to be smooth, compact manifold, without special mention. The manifolds are assumed to be homeomorphic to a sphere, otherwise the metric with  $K \geq 0$  does not exist. At the same time a part of the results will remain true in the case of compact manifolds with boundary. Moreover, so far as the systems under consideration are geodesical flows reduced from smooth Hamiltonian systems appearing in physical problems (where the conditions of smoothness are usually fulfilled) one can freely suppose that the results are true for this general case.

In Section 2 we have seen that the geodesics satisfy the Jacobi equation (2.3) having in two-dimensional case a form of (2.8):

$$z''(s) + K(s)z(s) = 0.$$

This equation substitutes Eqs (4.2). The Lyapunov number for function

is

$$\chi(\gamma(0), u) = \lim_{s \rightarrow +\infty} \frac{\ln(z^2 + z'^2)}{2s}, \quad (4.7)$$

where  $\gamma(s)$  is a geodesic with  $u = \frac{d}{ds} \gamma(0)$  with the Jacobi field  $\eta = z \cdot \hat{n}$ ,  $\|\hat{n}\| = 1$  defined on it. Here the trivial (zero) numbers are absent, and what is more important, this equation is easier to study.

Consider a closed geodesic on M  $\gamma(0) = \gamma(T)$ ; then  $K(s) = K(s+T)$  and one has Jacobi equation with periodical coefficients

$$z''(s) + K(s)z(s) = 0, \text{ where } K(s) = K(s+T).$$

It is known [26] that this equation has solutions of the following types ( $\varphi, \psi$  are periodical functions with a period T):

Hyperbolic:

$$z(s) = e^{\lambda s} \cdot \varphi(s) + e^{-\lambda s} \cdot \psi(s),$$

where  $\chi = \lambda$ , i.e. hyperbolic closed geodesics are stochastic;

Linear:

$$z(s) = \varphi(s) + s \psi(s),$$

i.e. geodesics are ordered;  $\chi = 0$ .

Elliptic:

$$z(s) = e^{i\alpha s} \varphi(s) + e^{-i\alpha s} \psi(s), \quad 0 < \alpha < \pi, \quad \chi = 0.$$

Consider a two-dimensional area transversal to the closed geodesic, beginning and returning on this area. The resulting transformation  $\mathcal{P}$  of the area on itself is called after Poincaré, and the obtained picture - Poincaré section. Evidently, the Jacobi field around the geodesic defines the linear part of  $\mathcal{P} - P$ . One can show [25] that if the geodesic is of hyperbolic type, then  $|Tr P| > 2$ ; if linear, then  $|Tr P| = 2$ ; if elliptic, then  $|Tr P| < 2$ . A closed geodesic is called twisting, if it is elliptical.  $\alpha \neq \frac{2\pi}{3}, \frac{2\pi}{4}$ , and satisfies a distinct condition on  $S^1$  [15, 27].

The null solution of Hamiltonian linear system is strongly stable, if it is stable and every null solution of close enough linear system is stable too [25].

It is proved (see [25]) that if  $|Tr P| < 2$ , then the null solution is strongly stable.

This result can be applied to an equation with nearly constant coefficients

$$\ddot{y}(t) + \omega_0^2 (1 + \varepsilon \sin(\omega_1 t)) y(t) = 0, \quad \varepsilon \ll 1. \quad (4.8)$$

Let us represent the system (4.8) as a point on a surface of parameters  $\varepsilon, \omega = \frac{\omega_0}{\omega_1}$ . For this system it is known [26, 25] that all points of the axis  $\omega$ , besides points  $\omega = \frac{m}{2}$  ( $m = 1, 2, \dots$ ) correspond to strongly stable solutions of Eq.(4.8) (Fig.2).

## 5. Integrable and Near-Integrable Systems

An N-dimensional system ( $N = \dim M$ ) is called integrable if N independent first integrals in involution are known (their Poisson brackets are zero: here and below for details, see [25]). Denote  $I_1, \dots, I_N$  integrals of motion. Consider the set of levels of functions  $I_j$ :

$$M_I = \{x; I_j(x) = I_j^0, \quad j = 1, \dots, N\}. \quad (5.1)$$

From Liouville theorem it follows that if  $M_I$  is compact and connected, then it is diffeomorphic to N-dimensional torus

$$T^N = \{(\vartheta_1, \dots, \vartheta_N) \text{ mod } 2\pi\} \quad (5.2)$$

and the phase flow for Hamiltonian function H defines a conditional-periodical motion on  $M_I$ , i.e. in angular coordinates  $\vartheta = (\vartheta_1, \dots, \vartheta_N)$

$$\frac{dI}{dt} = 0, \quad \frac{d\vartheta}{dt} = \omega(I),$$

where

$$\omega(I) = \frac{\partial H}{\partial I}.$$

Variables  $(I, \vartheta)$  are the so-called action-angle coordinates.

The motion on invariant torus  $I = \text{const}$  is conditional-periodical. Note that the frequencies  $\omega(I)$  depend on that which torus is under consideration, i.e. what values of first integrals are fixed. The case when the frequencies  $\omega(I)$  are functionally independent of N variables I is called non-degenerate case, i.e. when

$$\det \left| \frac{\partial \omega}{\partial I} \right| \neq 0. \quad (5.3)$$

Thus in non-degenerate case on different tori in phase space of unperturbed problem conditional-periodical motion with different number of frequencies is realized.

Let us see now what happens with non-resonant invariant tori at small perturbation of Hamiltonian function. According to Kolmogorov-Arnold-Moser (KAM) theorem, if unperturbed Hamiltonian system is non-degenerated, then at small enough conservative Hamiltonian perturbation the most of non-resonant invariant tori do not disappear, but become slightly distorted only. Therefore in phase space of perturbed system invariant tori exist, everywhere densely covered up by phase conditional-periodical trajectories, with frequencies equal to the number of degrees of freedom. These invariant tori form a majority in the sense that the measure of the supplement to their unification is small together with the perturbation.

#### 6. The Regularity of Systems with $K \geq 0$ : the Henon-Heiles Case

In this Section we shall study the stability of Henon-Heiles system; however the whole analysis can be repeated for arbitrary Hamiltonian system with non-negative curvature  $K$ , e.g. Contopoulos system, etc.

The Hamiltonian of Henon-Heiles has a well-known form [14]

$$H(p, q) = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + x^2 y - y^3/3, \quad (6.1)$$

where the cubic terms are smaller than the quadratic ones (the small  $E$ ).

Using the expression (2.2) we calculate the Riemannian curvature

$$K = \frac{(\nabla W)^2 - W \Delta W}{2W^3}; \quad (6.2)$$

$$W = E - \frac{x^2 + y^2}{2} - x^2 y + y^3/3.$$

First, note that  $\Delta W = -2$  and hence

$$K = \frac{(\nabla W)^2 + 2W}{2W^3} > 0$$

on  $M$ , where  $M = \{(x, y); W > 0\}$

So far as on the boundary of  $M$   $K \rightarrow +\infty$  ( $W = 0, (\nabla W)^2 \neq 0$ ) and  $M$  is compact, then  $K_{\min} > 0$  exists, so that

$$K \geq K_{\min}.$$

The curvature is equal to

$$K = \frac{E + 2(x^2 y - y^3/3)}{(E - \frac{x^2 + y^2}{2} - x^2 y + y^3/3)^3}. \quad (6.3)$$

Consider first (6.1) neglecting the cubic terms (a system of free oscillators):

$$H_0(p, q) = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2}. \quad (6.4)$$

For this system

$$K_0 = \frac{E}{(E - \frac{x^2 + y^2}{2})^3} \quad (6.5)$$

It is easy to see that

$$\begin{aligned} x_0 &= \sqrt{E} \cdot \cos t, \\ y_0 &= \sqrt{E} \cdot \sin t, \end{aligned} \quad (6.6)$$

$$K_0 = \frac{8}{E^2} = \text{const},$$

is a solution of this system.

The transformation of solution (6.6) for Henon-Heiles system (6.1) one can represent in a form ( $\sqrt{E}$  is a small parameter)

$$X = X_0 + E\tilde{x} + o(E),$$

$$y = y_0 + E\tilde{y} + o(E),$$

where  $X, y$  and  $\tilde{x}, \tilde{y}$  are the solutions of systems of equations

$$\ddot{\tilde{x}} = -\tilde{x} - 2xy,$$

$$\ddot{\tilde{y}} = -\tilde{y} - x^2 + y^2;$$

$$\ddot{\tilde{x}} = -\tilde{x} - \sin 2t,$$

$$\ddot{\tilde{y}} = -\tilde{y} - \cos 2t.$$

The second system has a solution

$$\tilde{x} = \frac{1}{3} \sin 2t,$$

$$\tilde{y} = \frac{1}{3} \cos 2t.$$

Thus we come to the partial solution of the system (6.1):

$$X = \sqrt{E} \cos t + \frac{E}{3} \sin 2t + o(E),$$

$$y = \sqrt{E} \sin t + \frac{E}{3} \cos 2t + o(E). \quad (6.7)$$

Using the expression

$$W = E - \frac{x^2 + y^2}{2} - x^2 y + y^3/3 = \frac{\dot{x}^2 + \dot{y}^2}{2}, \quad (6.8)$$

Eq.(6.3) can be rewritten in a form

$$K = \frac{E + 2(E - \frac{\dot{x}^2 + \dot{y}^2}{2})}{(\frac{\dot{x}^2 + \dot{y}^2}{2})^3}. \quad (6.9)$$

Substituting (6.7) into (6.9) we have

$$K = \frac{8}{E^2} (1 + 2\sqrt{E} \sin 3t + o(E)). \quad (6.10)$$

So far as the argument in Jacobi equation is affine parameter  $S$ , in Eq.(6.10) one should transform the time according to Eq.(4.5):

$$s = \int_0^t \sqrt{2} W(\tau) d\tau = \sqrt{2} \int_0^t (\frac{\dot{x}^2 + \dot{y}^2}{2}) d\tau$$

and with account of (6.8)

$$s = \frac{E}{\sqrt{2}} t + O(E^{3/2})$$

or

$$t = \frac{\sqrt{2}}{E} s + O(\sqrt{E}).$$

Finally for  $K$  we have

$$K(s) = \frac{8}{E^2} \left[ 1 + 2\sqrt{E} \sin\left(\frac{3\sqrt{2}s}{E}\right) \right]. \quad (6.11)$$

Using the condition of strong stability of null solution of Jacobi equation given in Section 4, from (6.11) we have

$$\frac{\omega_0}{\omega} = \sqrt{\frac{8}{E^2}} / \frac{3\sqrt{2}}{E} = \frac{2}{3} \neq \frac{m}{2}, \quad m = 1, 2, \dots$$

Thus, the obtained solution (6.7) of Henon-Heiles system (6.1) is strongly stable one.

This result explains the stability islands seen on Poincare sections obtained originally by Henon and Heiles by numerical methods [14]. In Fig.3 (from [14]) two points of strong stability calculated using Eq.(6.7) are shown.

## 7. Typical Properties of Dynamical Systems

Consider the space  $\Gamma^r(TM, \mu)$  of all dynamical systems of class  $C^r$  preserving the given measure  $\mu$ . A subspace of this space is massive if it contains everywhere a dense subset of  $G_\delta$  type - countable intersection of open sets; one can imagine it as a set of measure 1 in a space of measure 1 [28].

A property of dynamical systems is called typical if it is executed for the elements of massive subset in  $\Gamma^r(TM, \mu)$ . Our aim is to show that RS-systems are typical ones.

As it is shown in [17], if  $M$  is compact differentiable manifold and  $r \geq 4$ , then the following property of Riemannian metric  $g$  is typical in  $\Gamma^r(TM, \mu)$ :

Poincaré transformation  $P$  of every closed geodesics on a manifold  $M$  with metric  $g$  is hyperbolic or twisting.

In [17] it is supposed that in the general case of a manifold with finite fundamental group (e.g. connected manifold) it is optional all the closed geodesics to be of hyperbolic type. In other words, non-hyperbolic closed geodesics can exist; then the above-mentioned theorem guarantees those geodesics to be of twisting type.

Let us prove the following statement:

If  $f^t: M \rightarrow M$ ,  $\dim M = 2$  geodesic flow has a closed twisting geodesics, then this flow is non-ergodic, and ordered motion is realized in a region of positive measure.

Denote this geodesics by  $\gamma(t)$ ,  $\gamma(0) = \gamma(T) = 0$ . Local symplectic Poincaré transformation is defined on  $P: (R^2, 0) \rightarrow (R^2, 0)$ , which is twisting. According to KAM theory [25, 27], then a large set of invariant circles close to 0 from  $R^2$  exist, on each of them this transform-

ation acts as quasi-periodic one. Denote one of these circles by  $S^1$ , and

$$\partial A = \{f^t(S^1); 0 \leq t < T\}. \quad (7.1)$$

Evidently,  $\partial A$  is a tube around  $\gamma$ . Let  $A$  be the interior set of this tube, then

$$f^t A = A = f^{-t} A$$

and

$$0 < \mu(A) < 1,$$

i.e.  $f^t$  is non-ergodic (see (3.1)) and the regular trajectories have positive measure on  $M$ . Thus the existence of islands of ordered motion is a typical property of dynamical systems on connected compact manifolds.

Consider now the properties of systems in stochastic region. From [23] it follows that the condition  $h > 0$  is their typical property, hence non-zero Lyapunov numbers exist, and the region consists of ergodic components of positive measure. Our aim is to show that if  $K \geq 0$ , then cannot be Anosov U-system on these ergodic components. Note that there was an idea (see [30]), founded on numerical analysis, about the similarity of stochastic components and U-systems.

Before proceeding to the proof, it is useful to give some more definitions (see [13, 17]).

Let  $f^t: M \rightarrow M$  be a flow on a manifold  $M$  of class  $C^r$ ,  $r \geq 1$  with a given vector field  $X(x)$  ( $x$  from  $M$ ).

The flow  $f^t$  is called U-flow, if there exist two continuous distributions  $E^-$  and  $E^+$ , such that

1)  $T_x M = E_x^- \oplus E_x^+ \oplus Z_x$  for every  $x$  from  $M$ , where  $Z_x$  is induced by  $X(x)$ ;

2)  $d\dot{f}^t E_x^- = E_{\dot{f}^t(x)}^-$ ,  $d\dot{f}^t E_x^+ = E_{\dot{f}^t(x)}^+$  for every  $x$  from  $M$ ,  
and  $t$  from  $R$ ;

3) two constants  $c > 0$ ,  $\lambda > 0$  exist, such that for every  
from  $M$  and  $t > 0$

$$\|d\dot{f}^t v\| \leq c \cdot e^{-\lambda t} \|v\|, \text{ if } v \text{ is on } E_x^-;$$

$$\|d\dot{f}^t v\| \geq c^{-1} e^{\lambda t} \|v\|, \text{ if } v \text{ is on } E_x^+.$$

(7.2)

The subspaces  $E_x^-$  and  $E_x^+$  are called stable (compressing) and  
unstable (expanding), respectively.

Let us give some properties of U-flows that will be useful below.

1. Every geodesical flow  $\dot{f}^t$  on  $M$  having invariant stable and unstable  
fibers  $W^-$  and  $W^+$  so that  $\dim W^- + \dim W^+ = 2 \cdot \dim M - 2$   
is an U-flow [15].

2. If the geodesical flow  $\dot{f}^t$  on  $M$  is an U-flow, then there exist such  
 $t_0 > 0$ ,  $\lambda > 0$  ( $\lambda$  - from the definition of U-flows), that for  
every orthogonal Jacobi field with  $n(0) = 0$  [17]

$$\|n(t)\| \geq \|\nabla_u n|_{t=0}\| \operatorname{sh}(\lambda t), \quad t \geq t_0. \quad (7.3)$$

Now we prove the following statement for the flow  $\dot{f}^s: M \rightarrow M$ .  
 $\dim M = 2$ .

If  $K \geq 0$ , then  $\dot{f}^s$  cannot be an U-flow.

Assume that  $\dot{f}^s$  is an U-system, then such  $s_0 > 0$ ,  $\lambda > 0$   
must exist that  $z(0) = 0$  and

$$|z(s)| \geq |z'(0)| \operatorname{sh}(\lambda s), \quad s \geq s_0,$$

whence it follows that

$$|z(s)| \neq 0 \quad \text{at } s \geq s_0.$$

From Jacobi equation

$$z'' = -Kz;$$

$$z'' \leq 0 \quad \text{at } z > 0,$$

$$z'' \geq 0 \quad \text{at } z < 0,$$

$$|z(s)| \leq |z'(s_0)| \cdot s;$$

we have

$$|z'(s_0)| \cdot s \geq |z(s)| \geq |z'(0)| \operatorname{sh}(\lambda s), \quad (7.4)$$

$$|z'(s_0)| \cdot s \geq \operatorname{sh}(\lambda s) \quad \text{at } s \geq s_0,$$

i.e. we come to a contradiction.

Thus the considered systems in their stochastic regions cannot be  
U-systems.

Let us now show that geodesical flow  $\dot{f}^s: M \rightarrow M$ ,  $\dim M = 2$  at  
 $K \geq 0$  cannot have transversal fibers \*).

Assume that transversal fibers do exist. Then stable and unstable fibers

\*) A fiber is called transversal if it is invariant with respect to the  
dynamical system and is either stable or unstable, i.e. under the action  
of  $\dot{f}^s$  every tangential to the fiber vector either decreases or in-  
creases in length with exponential rate assumed to be uniform.

Dynamical system with transversal fiber is that having even one trans-  
versal fiber [31]:

$W^-$  and  $W^+$  for symmetrical in time (geodesical flow) exist, so that

$$\dim W^- + \dim W^+ = 2. \quad (7.5)$$

From the property 1 of U-systems formulated above it follows that our system with  $\dim M = 2$  must be an U-system, i.e. we come to a contradiction.

According to Sinai [21] the way to determine when a given dynamical system is a K-system lies through the construction and investigation of transversal fibers.

Hence one can conclude that due to the absence of transversal fibers the considered systems seem not to be K-systems.

An example of a billiard type system (representable as a geodesical flow with  $K \geq 0$ ) is known [33], whose stochastic regions are of mixing type, and this can be maximal property of those systems.

Thus almost all two-dimensional geodesical flows on a connected manifold have ordered regions. At stochastic regions consisting of several ergodic components with  $K \geq 0$ , the systems cannot be U-systems, have no transversal fibers and seem not to be K-systems.

## 8. Conclusion

In this paper we have studied some regular and stochastic properties of dynamical systems with non-negative curvature. The investigation of two-dimensional systems is reduced to the study of Jacobi equation with  $K \geq 0$ .

It is shown that the existence of positive-measured islands of regular and stochastic motion is the typical property of these systems, i.e. almost all of them are RS-systems. In stochastic regions consisting of ergodic components with positive entropy the systems have no transversal fibers.

At present we are not aware of a proof whether there exist K-systems

without transversal fibers, so we formulate our conclusion as a hypothesis: in stochastic regions the considered two-dimensional systems with  $K \geq 0$  cannot be K-systems.

A two-dimensional system with  $K \geq 0$  is the system of equations of Yang-Mills classical mechanics, mentioned in Introduction. In [32] are presented arguments that this system is K-system. Our analysis seems not to confirm this conclusion.

In view of the importance for the study of dynamics of spiral galaxies we have considered as an example the Henon-Heiles system. We have shown the existence of strongly stable solutions for it. This fact explains the existence of stability islands obtained earlier by means of numerical methods. The rich diversity of motions of stars in spiral galaxies becomes evident. These results are in agreement with those of papers [6,7], where by means of investigation of N-body gravitating system it is shown that disk-like systems (spiral galaxies) are significantly more regular than spherical ones (elliptical galaxies). The regularity of spiral galaxies at the present epoch of the evolution of the Universe is conditioned also by the suppression of Arnold diffusion for them [8,9].

Numerical study of stellar systems (N-body system) using the conceptions of ergodic theory is performed in [34] (see also [9]).

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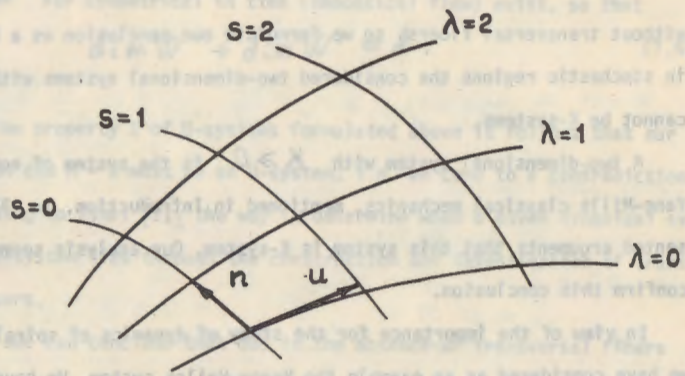


Fig.1

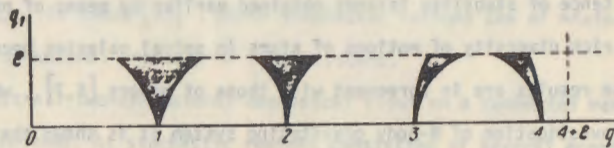


Fig.2

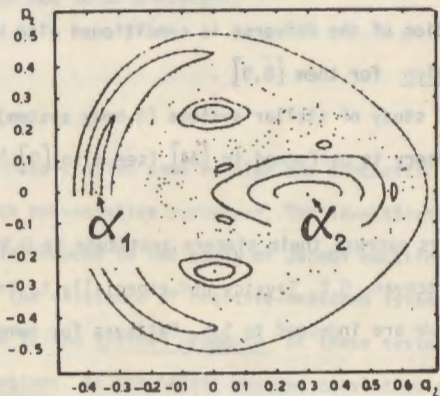


Fig.3

Figure Captions

Fig.1. Geodesics smoothly depending on parameter  $\lambda$  :  
velocity vector is  $u = d/ds$  and  $n = d/d\lambda$ .

Fig.2. Regions of strong stability ( $\text{Tr} P < 2$ )  
and instability (dashed) of solutions of Jacobi  
equations with nearly constant coefficients (4.8)  
(from [26]).

Fig.3. Poincaré sections for the Henon-Heiles model  
for  $E = 0.125$  [14] with marked points of strong  
stability  $\alpha_1 = (+0.39, 0)$ ,  $\alpha_2 = (-0.31, 0)$   
calculated using Eq.(6.7).

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**ТИПИЧНЫЕ СВОЙСТВА ДИНАМИЧЕСКИХ СИСТЕМ С НЕОТРИЦАТЕЛЬНОЙ  
КРИВИЗНОЙ**

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