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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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TO THE CONFINEMENT PROBLEM

ЦНИИатоминформ

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К ПРОБЛЕМЕ КОНФАЙНМЕНТА

В настоящей статье предполагается такая точка зрения на разделение физических величин на наблюдаемые и ненаблюдаемые, когда с последними связан эрмитовый оператор, для которого проблема собственных значений не разрешима.

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1. Introduction

Many problems in gauge field theory are left unsolved, in particular the confinement phenomenon is not so far explained satisfactorily. Why certain physical quantities are unobservable (being in the confinement phase), whereas the others constructed from those initial ones are observable? Thus, e.g. the Hamiltonian H_{YM}, being an observable physical quantity, is a function of canonically conjugated variables E_i^a and A_i^a which are unobservable.

This work proposes such a viewpoint for separation of physical quantities into observable and unobservable ones, when the latters are connected with the Hermitian operator for which the eigenvalue problem is unsolvable. Indeed, the only possible measurement results of physical quantities are eigenvalues of appropriate Hermitian operators. Therefore, if the quoted problem is unsolvable for them, then their measurement is impossible in any experiment.

Such interpretation is grounded on the fact, established by von Neumann and Stone in their study of unlimited Hermitian operators in Hilbert spaces, that there exist Hermitian operators for which the eigenvalue problem is unsolvable [1-3] , i.e. they have neither eigenvalues nor asymptotic states.

So we shall suppose that the Hermitian operators with solvable eigenvalue problem correspond to observable physical quantities, while those with unsolvable one* correspond to unobservable physical quantities, i.e. to those in the confinement phase. A success of such a program would consist in the proof that, say, the gluon field operators E_i^a and A_i^a were operators with unsolvable eigenvalue problem, whereas the operator H_{YM} with solvable one.

2. Conformal Quantum Mechanics

Let us show that already in a sufficiently simple field-theoretical model there emerge physical quantities being in the confinement phase in the above-mentioned sense.

Consider the massless scalar theory with self-interaction [6]

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - g\varphi^{\frac{2d}{d-2}} , \qquad (2.1)$$

where d is total space-time dimension. If $d = 1$, then (2.1) turns into a quantum-mechanical system with a Lagrangian

$$\mathcal{L} = \frac{1}{2} p^2 - g/x^2 , \quad g \geqslant 0 . \qquad (2.2)$$

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* It should be noted that the eigenvalue problem is always solvable for the limited Hermitian operators [2-4] ; however operators emerging in the quantum mechanics satisfy canonical commutation relations $[p, x] = -i$, being unlimited and defined only on everywhere dense subset D of Hilbert space $\mathcal{H}$ [5].

The action of this system is invariant under conformal group $O(2.1)$ with generators proportional to Hamiltonian H , dilatation $\mathcal{D}$ and conformal generator K with the algebra [6] :

$$[H, \mathcal{D}] = iH , \quad [K, \mathcal{D}] = iK , \quad [H, K] = 2i\mathcal{D} \qquad (2.3)$$

Operators H , $\mathcal{D}$ and K in the Schredinger representation have the form

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{g}{x^2} , \quad \mathcal{D}_{os} = \frac{1}{2}\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right), \quad K_{os} = \frac{x^2}{2} . \qquad (2.4)$$

Let us <u>calculate firstly the defect indices of operator H</u> (for details, see Appendix). Choose L^2 as a Hilbert space. H is defined on everywhere dense subset $D(H) = C_0^\infty(0, \infty)$ *.

Operator H is Hermitian one, and the one conjugated to it is equal to (see Appendix)

$$H^* \psi = -\frac{1}{2}\psi'' + g/x^2 \psi ,$$
$$D(H^*) = \left\{ \psi \in L^2, \ H^*\psi \in L^2 \right\} . \qquad (2.5)$$

Since the complex conjugation operation commutes with H , its defect indices are equal: $n_+ = n_-$ [8]. So long as equations for zero modes (A.3) may have two linearly independent solutions, the defect indices of H are (0.0), (1.1) or (2.2), i.e. H has zero-, one- and four-parametrical families of self-adjoint extensions.

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* The functions of this subset identically turn to zero in some vicinities of zero and infinity.

In order to understand which of these cases takes place, one should study the behaviour of solutions for zero-mode equation (A.3)

$$(H^{*} \pm i\mathbb{1})\Psi = 0 \qquad (2.6)$$

On infinity, from two solutions of (2.6) only one is quadratically integrable: $\exp\{(-1 \pm i)x/\sqrt{2}\}$. This case corresponds to the limiting point on infinity [8.] Near zero, the solutions of (2.6) have the form of x^{α}, where

$$\alpha_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + 2g}\ , \quad \alpha_2 = \frac{1}{2} - \sqrt{\frac{1}{4} + 2g}\ , \qquad (2.7)$$

therefore two cases are possible here:

i) $g < 3/8$ - both solutions of (2.6) are quadratically integrable near zero - this is the limiting circle at the origin.

ii) $g \geqslant 3/8$ - only one solution is integrable at the origin ($\alpha = \alpha_1$) - this is the limiting point at zero.

At ii) ($g \geqslant 3/8$) we have two limiting points at zero and infinity, therefore (2.6) has no solutions belonging to L^2, and $(n_+, n_-) = (0,0)$ operator H is self-adjoint:

$$D(H) = D(H^{*}), \qquad n_+ = n_- = 0\ . \qquad (2.8)$$

The spectrum of H is continuous and lies in the interval $(0, \infty)$.

Let us now clarify the behaviour of the other physical quantities in this region of interaction constants $g \geqslant 3/8$. The operators $K_n = x^n/n$ are self-adjoint. The zero modes of operator $\mathcal{D}_{08} \pm i\mathbb{1}$ (2.4) are in fact $\Psi_+ = c\,x^{-5/2}$ and $\Psi_- = c\,x^{3/2}$ and do not belong to $L^2(0,\infty)$: hence its defect indices are (0, 0) and it is self-adjoint as well. They all

have spectra and asymptotic states.

Consider now operators of the form

$$\mathcal{D}_n = x^n p + p x^n\ , \qquad n = 0, 2, 3 \ldots \qquad (2.9)$$

The defect indices of these operators are equal to (1, 0) since the equations $(\mathcal{D}_n^{*} \pm i\mathbb{1})\Psi = 0$ have solutions

$$\Psi_+^{(n)} = c\,x^{-\frac{n}{2}} e^{-\frac{1}{2(n-1)x^{n-1}}}\ , \quad \Psi_-^{(n)} = c^{-\frac{n}{2}} e^{\frac{1}{2(n-1)x^{n-1}}}\ , \qquad (2.10)$$

one of which does not belong to $L^2(0,\infty)$. Hence their eigenvalue problem is unsolvable, so they refer to unobservable quantities. Physically, this is due to the fact that because of strong repulsion, at the origin the "particle" moves only on the right (no tunneling transition).

At i) ($g < 3/8$) the defect indices of H are equal to (1, 1) and it has one-parametrical family of self-adjoint extensions which realizes by imposing more common boundary conditions at zero*

$$\Psi'(0) = -a\,\Psi(0) \qquad (2.11)$$

Now the "particle" can tunnel and be in the interval $(-\infty, +\infty)$. It is of interest to consider the defect indices of operators K_n and $\mathcal{D}_n$. In this region of interaction constants ($g < 3/8$ - weak repulsion at zero) the "particle" moves in the interval $(-\infty, +\infty)$, hence the defect indices are (0,0), all of them are observable and have asymptotic states.

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* At $a > 0$, there appears, in addition to continuous spectrum of H a discrete level with negative energy occurring due to delta-like attraction at zero.

3. Conclusion

The above-considered example of the quantum conformal mechanics illustrates the confinement viewpoint proposed in this paper. Realization of this programme for non-Abelian gauge theories is connected with difficulties since calculation of defect indices is a complicated problem, even for the finite-dimensional quantum mechanical systems.

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APPENDIX

Resolution of the Eigenvalue Problem

The solvability conditions of the eigenvalue problem are connected with the calculation of defect indices for the Hermitian operator A in the Hilbert space [1.2]. Recall some definitions from the Hermitian operators theory [5].

If on some everywhere dense range of definition $D(A)$ the operator A satisfies the relation (A.1)

$$(A\Psi,\varphi)=(\Psi,A\varphi) \tag{A.1}$$

for all Ψ and $\varphi\in D(A)$, then A is called Hermitian or symmetrical.

Let $D(A^*)$ be a set of φ from $\mathcal{H}$ for which there exists such Ψ from $\mathcal{H}$ that

$$(A\chi,\varphi)=(\chi,\Psi), \quad \chi\in D(A) \tag{A.2}$$

It is assumed that $A^*\varphi=\Psi$. Operator A^* is called conjugated to A. Practically, to obtain operator A^*, one should find all the pairs (φ, Ψ) which satisfy (A.2).

Operator A is called self-adjoint if $A=A^*$, i.e. iff A is Hermitian one and $D(A)=D(A^*)$.

The signif.. ance of the self-adjointness consists in the fact that namely for them is solvable the eigenvalue problem [2] and only for them can be built exponents giving one-parameter group of unitary operators $U=\exp\{iAt\}$ describing the quantum mechanical dynamics.

The defect indices are calculated as follows [2,5]. The space N_z defined as a zero-mode space of operator $A^* - z \cdot \mathbb{1}$

$$N_z = Ker\,(A^* - z \cdot \mathbb{1}) \,. \qquad (A.3)$$

has constant dimension in the upper (respectively in the lower) (z)-plane, therefore it is assumed that $z = \pm i$. The defect indices of A are equal to dimensions of N_z spaces at $z = \pm i$

$$n_\pm = \dim N_\pm = \dim\, Ker\,(A^* \pm i \mathbb{1}) \qquad (A.4)$$

Here $D(A^*) = D(A) \oplus N_+ \oplus N_-$.

The von Neumann theorem reads: if

$n_+ = n_- = 0$, then A is self-adjoint;

$n_+ = n_- = n$, then there exists n^2 -parametrical extension of A up to self-adjoint one;

$n_+ \neq n_-$, then A has no self adjoint extension and the eigenvalue problem is unsolvable for it.

Thus, in order to define whether a given physical quantity is observable or not, it is necessary to calculate defect indices of its Hermitian operator.

It should be noted that though $n_+ \neq n_-$, the Hermitian operator of A is well-defined and can "participate" in the theory formulation.

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