Preprint ЕФИ-846(73)-85

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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

O.M.KHUDAVERDYAN, R.L.MKRTCHYAN, L.A.ZURABYAN

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ЦНИИатоминформ

ЕРЕВАН-1985

Препринт ЕФИ-846(73)-85

Л.А.ЗУРАБЯН, Р.Л.МКРТЧЯН, О.М.ХУДАВЕРДЯН

ОБ АКСИАЛЬНЫХ АНОМАЛИЯХ ВО ВНЕШНИХ ТЕНЗОРНЫХ ПОЛЯХ

Рассмотрен вопрос о вычислении аномалии аксиального тока дираковских фермионов во внешних антисимметричных тензорных полях нечетного ранга. Представлена последовательность одномерных суперсимметричных моделей, суперзаряды которых есть, после квантования, операторы Дирака во внешних тензорных полях, а их виттеновская статсумма есть аномалия аксиального тока. Показано, что действие в соответствующем континуальном интеграле для статсуммы отличается от классического. Квазиклассическое приближение дает аномалию только в случае внешнего тензорного поля третьего ранга с нулевой внешней производной, и она вычислена в этом случае в произвольной размерности. Обсуждается интерпретация этого поля как внешнего гравитационного поля с кручением и связь с результатами Виттена и Альвареца-Гоуме и теоремой Атьи-Зингера.

Ереванский физический институт

Ереван 1985

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O.M.KHUDAVERDYAN*, R.L.MKRTCHYAN, L.A.ZURABYAN

ON THE AXIAL ANOMALIES IN EXTERNAL TENSOR FIELDS

Computation of the axial anomaly for Dirac fermions in external tensor fields is studied. The sequence of the supersymmetric one-dimensional models is presented. Their supercharges are equal, after quantization, to Dirac operators in external tensor fields, and the density of Witten's partition function gives the anomaly. It is shown, that action in the corresponding path integral differs from the classical one. Gaussian approximation gives the anomaly only in the case of third-rank tensor with zero exterior derivative and in that case anomaly is calculated in all dimensions. The interpretation of that field as the torsion of gravitational field and also connection with the results of Witten and Alvarez-Gaume and Atiyah-Singer index theorem are discussed.

Yerevan Physics Institute

Yerevan 1985

* Yerevan State University

Introduction.

The present work is devoted to the calculation of the axial current anomaly in the external antisymmetric tensor fields, in the (Euclidean) space-time with arbitrary (even) dimension d. We shall use the new method developed recently in [1-3], and our aim is also to analyze this method. The method is shown to allow one to calculate the anomaly actually only in the case of third-rank tensor with zero exterior derivative.

Axial anomaly, i.e. nonconservation of axial current in the quantum theory, has been discovered in the works of Schwinger [4], Adler [5], and Bell and Jackiw [6]. Namely, in the massless QED axial current

$$J^5_\mu = \bar{\Psi} \gamma_\mu \gamma^5 \Psi$$

is conserved classically, but after gauge-invariant quantization its divergence turns out non-zero [5,6]:

$$\partial_\mu J^5_\mu = -\frac{e^2}{4\pi} \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} .$$

Actually, anomaly is given entirely by the one-loop diagrams [7], which are generated by the Lagrangian of fermions in the external (non-quantized) field A_μ.

Let's consider the following Lagrangian describing the fermions, in space-time of dimension $d = 2n$, in the external matrix field $C(x)$

$$\mathcal{L} = \bar{\psi}\,(i\hat{\partial} + C)\,\psi = \bar{\psi}\,\mathcal{D}\,\psi\,, \tag{1}$$

where $\{C, \gamma^5\} = 0$ (by the γ^5 we denote the matrix $i^{\frac{d}{2}}\gamma_1 \cdot \ldots \gamma_d$). The field $C(x)$ has the following decomposition over the antisymmetrized products of the γ_μ -matrices:

$$C(x) = A_\mu(x)\gamma_\mu + i A_{\mu\nu\lambda}\gamma_{\mu\nu\lambda} + \ldots + i^{\frac{d-2}{2}} A_{\mu_1 \ldots \mu_{d-1}}\gamma_{\mu_1 \ldots \mu_{d-1}}$$

$$\gamma_{\mu\nu\lambda} = \gamma_{[\mu}\gamma_\nu \ldots \gamma_{\lambda]} \qquad \{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}\,,$$

where $A_{\mu_1 \ldots \mu_k}$ is antisymmetric tensor field of rank K.
$A_\mu(x)$ is the usual U(1) gauge field, and its presence provides that the Lagrangian (1) is invariant under the gauge transformations

$$\delta C = -\hat{\partial}\alpha(x)\,,\ \delta\psi = i\,\alpha(x)\,\psi\,,\ \delta\bar{\psi} = \bar{\psi}\cdot(-i\,\alpha(x))\,, \tag{2}$$

The axial vector field $B_\mu = \varepsilon_{\mu\mu_1 \ldots \mu_{d-1}} A_{\mu_1 \ldots \mu_{d-1}}$ is dual to $A_{\mu_1 \ldots \mu_{d-1}}$, and (1) is also invariant under local axial transformations

$$\delta B_\mu = -\partial_\mu \beta(x)\,,\quad \delta\psi = \gamma^5\psi\,\beta(x)\,,\quad \delta\bar{\psi} = \bar{\psi}\gamma^5\beta(x).$$

Consequently, both vector ($J_\mu = \bar{\psi}\gamma_\mu\psi$) and axial-vector ($J^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi$) currents are conserved classically in this model. The question under study is: how will the equation of the axial vector current conservation ($\partial_\mu J^5_\mu = 0$) be modified, if we quantize theory (1) in the vector gauge invariant (i.e. keeping $\partial_\mu J_\mu = 0$ after quantization) way?

The answer for $\partial_\mu J^5_\mu$ is the following [8,9,10,1]. Let's consider the kernel of the operator $\exp\left(-\frac{t}{2}\,\mathcal{D}^2\right)$

$$\psi(t,x) = \langle x|\exp\left(-\tfrac{t}{2}\not{D}^2\right)|x\rangle .$$

It is a matrix with spinor indices. At $t \to 0$ $\psi(t,x)$ has the expansion [11,12,8] :

$$\psi(t,x) \cong \sum_{k \le \frac{d}{2}} \psi_k(x) \left(\frac{t}{2}\right)^{-k} \tag{3}$$

It is convenient in what follows to use the definition

$$\varphi(t,x) = tr\,\gamma^5 \psi(t,x)$$

where the trace goes over spinor indices.

$\psi_k(x)$ are the Seeley coefficients. Anomaly is given by ψ_0:

$$\partial_\mu J^5_\mu = 2\, tr\,\gamma^5 \psi_0(x) .$$

The new method for the calculation of $tr\,\gamma^5 \psi_0(x)$, developed in [illegible] is the following. Consider the "Witten index"

$$Sp\,\gamma^5 e^{-\frac{t}{2}\not{D}^2} = \int dx\; tr\,\gamma^5 \psi(t,x) .$$

This is, up to γ^5, partition function for the quantum mechanics with hamiltonian $\frac{1}{2}\not{D}^2$. It has the path integral representation (derivation see in Sect. 1):

$$Sp\,\gamma^5 e^{-\frac{t}{2}\not{D}^2} = \int \mathcal{D}\psi_\mu\, \mathcal{D}x_\mu\; e^{-S(x,\psi)} .$$

$$S(x,\psi) = \int_0^t ds\; L\left(\psi_\mu(s), x_\mu(s)\right) . \tag{4}$$

Here $\psi_\mu(s)$ are real grassmanian anticommuting variables, which become γ_μ -matrices after quantization. Integration over ψ_μ is implied by the rules of Berezin [13] . The fields $\psi_\mu(s)$, $x_\mu(s)$ in (4) have periodic boundary conditions

$$x_\mu(0) = x_\mu(t), \quad \psi_\mu(0) = \psi_\mu(t).$$

The main contribution in the integral (4) at $t \to 0$ is given by the gaussian approximation: integral over constant fields $x_\mu(s) = x_\mu^0$, $\psi_\mu(s) = \psi_\mu^0$, and the integral over deviations from x_μ^0 and ψ_μ^0 in the gaussian approximation (exact treatment see in Sect. 1). Usually this main contribution is O(1), i.e. independent of t. It means that in those cases the expansion of $\varphi(t, x)$ at $t \to 0$ starts from the t-independent term being then followed by positive powers of t.

In that case gaussian approximation directly gives the anomaly $\partial_\mu J_\mu^5$. But, as will be shown below, in the general case, expansion of $\varphi(t, x)$ at $t \to 0$ starts from the negative powers of t and consequently anomaly is given not by the gaussian approximation of (4) but by the definite order of perturbation expansion near the gaussian approximation. In particular, this is the case for the anomaly in the external tensor fields. The only exception is the case of the third-rank tensor $A_{\mu\nu\lambda}$ with the zero exterior derivative

$$F_{\mu\nu\lambda\rho} = \partial_{[\mu} A_{\nu\lambda\rho]} = 0.$$

In this case anomaly is given by the gaussian approximation. The answer - (12) - is calculated by us in arbitrary dimension, and its discussion from the viewpoint of Atiyah-Singer theory is given in Sect. 2.

There is an interesting connection between the action in (4) in this particular case just discussed and recently discovered new two-dimensional nonlinear supersymmetric σ-models with Wess-Zumino term [14,15]. Namely, the action S_{cl} in (4) is the reduction of the action of those σ-models to dimension one. These models may be considered [14,15] as nonlinear

σ-models on the manifolds with non-Riemannian connections (with torsion) Such an interpretation for our model will be discussed in Sect. 2.

Below we shall consider the case, when only one of the tensors A_μ, $A_{\mu\nu\lambda}$ ...., is non-zero, i.e. we won't consider the mixed anomalies.

Now we shall discuss the action $S(x,\psi)$ entering in the path integral representation (4).

Take the following action

$$S_{cl}(x,\psi) = \int_0^t ds \Big[\frac{\dot{x}^2}{2} + \frac{i}{2}\psi\dot{\psi} + (2i)^k \Big((2k+1) A_{\mu_1 \dots \mu_{2k+1}} \dot{x}_{\mu_1} \psi_{\mu_2} \dots \psi_{\mu_{2k+1}}$$

$$- i F_{\mu_1 \dots \mu_{2k+2}} \psi_{\mu_1} \dots \psi_{\mu_{2k+2}} \Big) \Big] \tag{5}$$

$$F_{\mu_1 \dots \mu_{2k+2}} = \partial_{[\mu_1} A_{\mu_2 \dots \mu_{2k+2}]} .$$

This action is supersymmetric (under periodic boundary conditions), with supersymmetry transformations

$$\delta_\varepsilon x_\mu = i\varepsilon \psi_\mu \quad , \quad \delta_\varepsilon \psi_\mu = -\varepsilon \dot{x}_\mu \tag{6}$$

Supersymmetry algebra can be easily verified:

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \text{[illegible]}$$

Denoting the supersymmetry generator by Q and Hamiltonian corresponding to the action (5) by H, we may rewrite the previous equation as

$$\{Q,Q\} = 2H, \tag{7}$$

where in the l.h.s. the Poisson bracket in the presence of the anticommuting variables is used. This bracket is introduced in [16].

Q may be expressed through canonical variables [illegible]

$$Q = p_\mu \psi_\mu - (2i)^k A_{\mu_1 \dots \mu_{2k+1}} \psi_{\mu_1} \dots \psi_{\mu_{2k+1}}$$

Quantization of (5) leads to operators $\hat{x}_\mu, \hat{p}_\mu, \gamma_\mu$, the latter being a quantum version of the classical variable $\sqrt{2}\,\psi_\mu$ and satisfying the Dirac matrices relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The quantum version of generator Q is the previously considered Dirac operator in the external tensor field:

$$Q \to -\frac{\mathcal{D}}{\sqrt{2}}, \quad \mathcal{D} = i\hat{\partial} + i^\kappa A_{\mu_1 \dots \mu_{2\kappa+1}} \gamma_{\mu_1 \dots \mu_{2\kappa+1}} .$$

After quantization, including appropriate ordering of non-commuting operators, Hamiltonian H satisfies the quantum analog of (7), and one may say that quantization of (5) leads to the Hamiltonian $H = \frac{1}{2}\mathcal{D}^2$. Hence, one may expect that action $S(x, \psi)$ in (4) coincides with (5). Exact calculation (Sect. 2) using Berezin's method of path-integral representation of operators symbols [16,17] shows that action in (4) differs from (5), and the difference (in the case of the third-rank tensor $A_{\mu\nu\lambda}$ with zero exterior derivative) is

$$S_{qu} = -3\int_0^t ds\; A_{\mu\nu\lambda}(x(s)) A_{\mu\nu\lambda}(x(s)) , \tag{8}$$

This quantum addition to classical action in the path-integral representation is analogous to the term R/6 found by De Witt [18] in the path-integral representation of the propagator of scalar particle in the external gravitational field. (8) may also be interpreted as integral of the scalar curvature, but calculated from the non-Riemannian connection equal to $3A_{\mu\nu\lambda}$.

In Sect. 2 we discuss the connection of the results of Sect. 1 with the Atiyah-Singer index theorem, and also some statements of Sect. 1 are directly verified through the calculation of Seeley coefficients, In the Conclusion the results of the present work are briefly formulated. Appendix contains some formulae of the operators symbols calculus.

1. The Path Integral Derivation of the Axial Anomaly.

Our purpose is to get the formula (4), particularly the action $S(x,\psi)$ in this formula.

We shall use the method of Berezin [17], when instead of representing the operator kernel as a path integral, as usual (see, e.g. [19]), we represent its symbol. This method is the only possible if there are γ_μ - matrices in operator, as in $\mathcal{D}^2$.

The notion of the operator symbol and the treatment rules [16,17] , particularly the formulae for the calculation of the operators' product symbol through the symbols of multipliers, see in Appendix.

Using Weyl symbol for variables p_μ and x_μ (momenta and coordinates) and the generalization of the Weyl symbol notion for the case of the matrix operators one can obtain the representation of the symbol of the operator $\exp(-\frac{t}{2}\mathcal{D}^2)$ as a path integral by means of the following procedure [16].

Represent the operator $\exp(-\frac{t}{2}\mathcal{D}^2)$ as a product

$$e^{-\frac{t}{2}\mathcal{D}^2} = e^{-\frac{t}{2N}\mathcal{D}^2} \cdot \ldots \, e^{-\frac{t}{2N}\mathcal{D}^2} \qquad \text{(N multipliers)}$$

Turning to symbols

$$\mathrm{Symb}\, e^{-\frac{t}{2}\mathcal{D}^2} = \mathrm{Symb}\, e^{-\frac{t}{2N}\mathcal{D}^2} * \ldots * \mathrm{Symb}\, e^{-\frac{t}{2N}\mathcal{D}^2} =$$

$$= \int \prod_{i=1}^{N} d\xi_i\, d\psi_i\, dz'_i\, dz_i \exp\Big\{ \sum_{j=1}^{N} \Big[2i\big(z_j \omega z'_j + z'_j \omega z_{j+1} + z_{j+1}\omega z_j\big) + 2\big(\xi_j \psi_j + \psi_j \xi_{j+1} + \xi_{j+1}\xi_j\big) - \frac{t}{2N} H(z_j, \psi_j)\Big]\Big\},$$

where $\xi_{N+1} = \xi$, $z'_{N+1} = z$.

Integrating over ξ_j, z'_j we obtain

$$\mathrm{Symb}\, e^{-\frac{t}{2}\mathcal{D}^2}(\xi, z) = \int \prod_{i=1}^{N} d\psi_i\, dz_i \cdot$$

$$\exp\Big\{ 2i \sum_{k=1}^{\frac{N}{2}-1} \big[z'_{2k}\,\omega\,(z_{2k} - z_{2k-1}) - i\,\xi_{2k}(\psi_{2k} - \psi_{2k-1})\big] -$$

$$- \sum_{j=1}^{N} \frac{t}{2N} H(z_j, \psi_j) + 2i\,(z_N\,\omega\, z + z\,\omega\, z'_N) + 2(\psi_N \xi + \xi\,\xi_N)\Big\}$$

where

$$z'_{2k} = z_1 + \sum_{i=1}^{k-1} (z_{2i+1} - z_{2i}),$$

$$\xi_{2k} = \psi_1 + \sum_{i=1}^{k-1} (\psi_{2i+1} - \psi_{2i}),$$

where $H(z,\psi)$ is the symbol of operator $\mathcal{D}^2/2$

$$H(z,\psi) = \frac{1}{2} p_\mu^2 - 6i A_{\mu\nu\lambda}\, p_\mu \psi_\nu \psi_\lambda - 2F_{\mu\nu\lambda\rho}\, \psi_\mu \psi_\nu \psi_\lambda \psi_\rho -$$

$$- 18 A_{\mu\nu\lambda} A_{\mu\rho\sigma}\, \psi_\nu \psi_\lambda \psi_\rho \psi_\sigma + 3 A_{\mu\nu\lambda} A_{\mu\nu\lambda},$$

$$z = (p_\mu, x_\mu).$$

We write out this formula and the following ones only for the third-rank tensor, because we shall not need them in the general case. Besides, we omit all the normalization coefficients in the path integrals. We shall show in the end of this Section how to restore the true norms.

After transition to the limit $N \to \infty$ we have

$$\mathrm{Symb}\, e^{-\frac{t}{2}\mathcal{D}^2}(\xi, z) = \int \mathcal{D}\psi\, \mathcal{D}z\, \exp\Big\{ \int_0^t ds \Big[\frac{i}{2} z\,\omega\,\dot z +$$

$$- \frac{1}{2}\psi\dot\psi - H(z,\psi)\Big] - i\,[z\,\omega\, z(t) + z(0)\,\omega\, z + z(t)\,\omega\, z(0)]$$

$$- [\xi\,\psi(0) + \psi(0)\,\psi(t) + \psi(t)\,\xi]\Big\}.$$

Using equation

$$Sp\gamma^5 e^{-\frac{t}{2}\not{D}^2} = \int d\xi_\mu\, dp_\mu\, dx_\mu\, \mathrm{Symb}\, e^{-\frac{t}{2}\not{D}^2}(\xi, p, x)$$

after integration over ξ_μ, p_μ, x_μ we get inside the integral the expression $\delta(x(t)-x(0))\,\delta(p(t)-p(0))\,\delta(\psi(t)-\psi(0))$ and Gaussian integration gives finally

$$Sp\gamma^5 e^{-\frac{t}{2}\not{D}^2} = \int \mathcal{D}\psi_\mu\, \mathcal{D}x_\mu\, \exp\Big\{-\int_0^t ds\Big(\frac{\dot{x}^2}{2} + \frac{i}{2}\psi\dot{\psi} + 6i A_{\mu\nu\lambda}\dot{x}_\mu\psi_\nu\psi_\lambda + 2F_{\mu\nu\lambda\rho}\psi_\mu\psi_\nu\psi_\lambda\psi_\rho - 3A_{\mu\nu\lambda}A_{\mu\nu\lambda}\Big)\Big\}, \tag{9}$$

where we integrate over ψ_μ and x_μ with periodic boundary conditions

$$x_\mu(0) = x_\mu(t)\ , \quad \psi_\mu(0) = \psi_\mu(t).$$

It was noted in Introduction that the action in the path integral consists of two parts

$$S = S_{cl} + S_{qu}\ , \quad S_{qu} = -3\int_0^t ds\, A_{\mu\nu\lambda}A_{\mu\nu\lambda}$$

Defined by formula (5) term S_{cl} is invariant under the supersymmetry transformations, and the corresponding Hamiltonian H is equal, after quantization and proper solution of the problem of ordering of non-commuting multipliers, to supercharge Q

$$H = Q^2\ , \qquad Q = -\frac{\not{D}}{\sqrt{2}}.$$

Note that the ordering problem does not occur in Q .

The term S_{qu} in the action corresponds to our choice of the Hamiltonian. Note that it breaks the supersymmetry of $S_{cl} + S_{qu}$ under the transformations (6), although the theory (quantum mechanics) with $H = \frac{1}{2}\not{D}^2$ is supersymmetric (it possesses conserving supercharge Q).

We have noted earlier, that we need the coefficient $\varphi_3(\lambda)$ from the expansion ($t \to 0$)

$$\mathrm{Sp}\,\gamma^5 e^{-\frac{t}{2}\mathcal{D}^2} \simeq \int dx\, \mathrm{tr}\,\gamma^5 \sum_{k \le \frac{d}{2}} \varphi_k(x) \left(\frac{t}{2}\right)^{-k}$$

to get the anomaly

$$\partial_\mu J^5_\mu = 2\,\mathrm{tr}\,\gamma^5 \varphi_0(x)$$

Consider the following perturbation-theory expansion of the integral (9). Introduce the Fourier coefficients

$$x_\mu(s) = \sum_{n=-\infty}^{\infty} x^n_\mu e^{2\pi i n s/t} = x^0_\mu + \bar{x}_\mu(s) \;,\; x^0_\mu \;(x^0_\mu)^* \;\; x^n_\mu = (x^{-n}_\mu)^*$$

$$\psi_\mu(s) = \sum_{n=-\infty}^{\infty} \psi^n_\mu e^{2\pi i n s/t} = \psi^0_\mu + \bar{\psi}_\mu(s) \;,\; \psi^0_\mu = (\psi^0_\mu)^*, \; \psi^n_\mu = (\psi^{-n}_\mu)^*.$$

Write (9) in the following form

$$\mathrm{Sp}\,\gamma^5 e^{-\frac{t}{2}\mathcal{D}^2} = \int dx_0\, d\psi_0 \int \mathcal{D}\bar{x}\, \mathcal{D}\bar{\psi}\, \exp\Big\{ - S(x_0,\psi_0) -$$

$$- \int_0^t ds \Big[\frac{\dot{\bar{x}}^2}{2} + \frac{i}{2}\bar{\psi}\dot{\bar{\psi}} + 6i\,(2A_{\mu\nu\lambda}(x_0)\,\dot{\bar{x}}_\mu \bar{\psi}_\nu \psi^0_\lambda +$$

$$+ \partial_\alpha A_{\mu\nu\lambda}(x_0)\,\dot{\bar{x}}_\mu \bar{x}_\alpha \psi^0_\nu \psi^0_\lambda + \ldots) + \ldots \Big] \Big\},$$

where we expand all the functions in powers of $\bar{x}, \bar{\psi}$, e.g.

$$A_{\mu\nu\lambda}(x) = A_{\mu\nu\lambda}(x_0) + \partial_\alpha A_{\mu\nu\lambda}(x^0)\,\bar{x}^\alpha + \ldots$$

Keeping the terms $\int_0^t ds \left(\frac{\dot{\bar{x}}^2}{2} + \frac{i}{2}\bar{\psi}\dot{\bar{\psi}} \right)$ in the exponent, consider all the rest as a perturbation and expand the exponent

$$\varphi(t\, x^0) = \int d\psi^0 \int \mathcal{D}\bar{x}\,\mathcal{D}\bar{\psi}\; e^{\int ds \left(\frac{\dot{\bar{x}}^2}{2} - \frac{i}{2}\bar{\psi}\dot{\bar{\psi}}\right)} \left(1 + \ldots \right) = \qquad (10)$$

$$= \int d\varphi_0 \ \mathrm{sdet}\, \Phi \cdot (1 + \ldots)$$

The remaining in the exponent quadratic form (the free action) is designated by Φ , the perturbation theory expansion is substituted by dots. Taking into account that $\mathrm{sdet}\, \Phi \sim t^{-\frac{d}{2}}$, we ask ourselves: what terms of the perturbation theory are of the order of O(1)?

The consideration of the integral over φ° is crucial: one can easily verify its saturation, and non-saturating (having φ° raised to degrees less than d) terms of the perturbation theory give zero.

An important statement is the following. In the case of the third-rank tensor with zero exterior derivative

$$F_{\mu\nu\lambda\rho} = \partial_{[\mu} A_{\nu\lambda\rho]} = 0 \tag{11}$$

only the quadratic over $\bar{\chi}, \bar{\varphi}$ terms of the action $S_{c\epsilon}$ give the contribution in the order O(1) in (10)*). If (11) does not take place or in the action there is a tensor of a higher rank, then the terms of a higher than second order over $\bar{\chi}, \bar{\varphi}$ also give contribution in the order O(1) in (10).

There is a principal difference between these two cases. In the first case calculations can be done up to the end, because only Gaussian integral is to be calculated. In the second case the higher the space dimension is, still farther terms of the perturbation theory contribute, so to get the answer for arbitrary dimension is impossible.

---

*) The term S_{q4} gives no contribution in the order O(1).

Let's explain this statement in the third-rank tensor case. The action for it was shown earlier. Since in (10) $sdet\,\Phi \sim t^{-\frac{d}{2}}$, then terms of the perturbation theory giving contribution in the order O(1) are to be $\sim t^{\frac{d}{2}}$ for integral (10). One can easily indicate the non-quadratic over $\bar{x}, \bar{\varphi}$ terms of such an order, saturating the integral over φ^0, e.g. $t\,F\varphi^0\varphi^0\varphi^0\varphi^0\,\partial A\,\bar{x}\dot{\bar{x}}\bar{\varphi}\bar{\varphi}$ when d=4 and many other terms. We take here into account that the fluctuations of $\bar{x}$ are of the order of $t^{\frac{d}{2}}$, as can be seen from the free part of the action.

On the other hand, if $F_{\mu\nu\lambda\rho} = 0$ (11), then every φ^0 is multiplied by either $\bar{x}$ (in quadratic part) or $\bar{x}^2$ and higher degrees (in the other terms). That is why the saturation of the integral over φ^0 gives the degree $(\bar{x})^d$, i.e. $\sim t^{\frac{d}{2}}$, if the vertices from quadratic part are used, and an order more than $t^{\frac{d}{2}}$, if other terms of the expansion in $\bar{x}, \bar{\varphi}$ are used.

We shall calculate now the axial anomaly in external third-rank tensor field with zero exterior derivative. As discussed earlier, it is given by Gaussian integral

$$\int d\varphi^0 \int \mathcal{D}\bar{x}\,\mathcal{D}\bar{\varphi}\, \exp\Big\{-\int_0^t ds\Big(\frac{\dot{\bar{x}}^2}{2} + \frac{i}{2}\bar{\varphi}\dot{\bar{\varphi}} + 12 i A_{\mu\nu\lambda}\dot{\bar{x}}_\mu \bar{\varphi}_\nu \varphi^0_\lambda$$

$$+ 6i\,\partial_\alpha A_{\mu\nu\lambda}\,\bar{x}^\alpha \dot{\bar{x}}^\mu \varphi^0_\nu \varphi^0_\lambda\Big)\Big\}.$$

Passing to the integration over the Fourier coefficients x^n_μ, φ^n_μ, we get

$$\int d\varphi^0 \int \prod_{n=1}^{\infty} dx^n dx^{n*} d\varphi^n d\varphi^{n*} \exp\Big\{-\sum_{n=1}^{\infty}\Big[\frac{4\pi^2 n^2}{t^2} x^n_\mu x^{n*}_\mu -$$

$$-2\pi n\,\varphi^{n*}_\mu \varphi^n_\mu + 24\pi n\,\partial_{[\mu}A_{\nu]\alpha\lambda}\,\varphi^0_\alpha \varphi^0_\lambda x^{n*}_\mu x^n_\nu +$$

$$+ 24\pi n\, A_{\mu\nu\lambda}\varphi^0_\lambda\left(x^{n*}_\mu \varphi^n_\nu - x^n_\nu \varphi^{n*}_\mu\right) =$$

$$
= \int d\psi^0 \prod_{n=1}^{\infty} \mathrm{sdet}^{-1} \begin{pmatrix} \delta_{\mu\nu} + \frac{6t}{\pi n} \partial_{[\mu} A_{\nu]\lambda\rho} \psi_\lambda^0 \psi_\rho^0 \,, & \frac{6t}{\pi n} A_{\mu\nu\lambda} \psi_\lambda^0 \\ \frac{6t}{\pi n} A_{\mu\nu\lambda} \psi_\lambda^0 \,, & -\frac{t}{2\pi n} \delta_{\mu\nu} \end{pmatrix}
$$

$$
= \int d\psi^0 \prod_{n=1}^{\infty} \det{}^{-1} \left(\delta_{\mu\nu} + \frac{t}{\pi n} R_{\lambda\rho\mu\nu} \psi_\lambda^0 \psi_\rho^0 \right)
$$

$$
R_{\mu\nu\lambda\rho} = 3\partial_\mu A_{\nu\lambda\rho} - 3\partial_\nu A_{\mu\lambda\rho} + 72 A_{\mu\lambda\alpha} A_{\nu\alpha\rho} \,.
$$

We used here the equation

$$
R_{\mu\nu\lambda\rho}(A) = R_{\lambda\rho\mu\nu}(A)
$$

immediately following from Eq.(11).

Finally, skew-diagonalizing the matrix $\hat{R}_{\mu\nu} = R_{\lambda\rho\mu\nu} \psi_\lambda^0 \psi_\rho^0$

$$
\hat{R}_{\mu\nu} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_{d/2} \\ & & & -\lambda_{d/2} & 0 \end{pmatrix}
$$

we get

$$
\partial_\mu J_\mu^5 = 2 \frac{1}{(2\pi t)^{d/2}} \int d\psi^0 \prod_{i=1}^{d/2} \frac{t\lambda_i}{\mathrm{sh}\, t\lambda_i} \tag{12}
$$

Here λ_i are the eigenvalues of the matrix $\hat{R}_{\mu\nu}$.

The normalization in (12) can be obtained in the following way. The first term of expansion of the expression under the integral over ... coincide to the first term of the expansion (3) which is ...

In order to express the anomaly in a given dimension $d=2n$ through the curvature tensor, one must expand the right-hand side of (12) in Taylor series and pick out the powers $d/2$ in λ_i (for only they contribute to the integral over ψ^0). The sum of these terms is symmetric function of λ_i and it can be written as a linear combination of the products of the $\hat{R}_{\mu\nu}$-matrix powers traces. After that, the integration over ψ_0 is trivial and gives tensor $\varepsilon_{\mu_1\ldots\mu_d}$. E.g., when d = 4 . the terms of the second power in λ in the expansion of (12) equal to

$$\int d\psi^0 \frac{1}{6}(\lambda_1^2-\lambda_2^2) = \frac{1}{12} R_{\mu\nu\alpha\beta} R_{\lambda\rho\alpha\beta} \varepsilon_{\mu\nu\lambda\rho}$$

and

$$\partial_\mu J_\mu^5 = \frac{1}{24\pi^2} R_{\mu\nu\alpha\beta} R_{\lambda\rho\alpha\beta} \varepsilon_{\mu\nu\lambda\rho}$$

It is obvious that the anomaly (12) is nonzero only when

2. Atiyah-Singer Theorem. Seeley Coefficients.

The expression (12) for anomaly in external gravitational field is similar to one obtained in [1]. Let us recall the results of [1], where the Dirac operator $\mathcal{D}(\omega(e))$ in the external gravitational field is considered:

$$\mathcal{D}(\omega(e)) = i\, e_a^\mu \gamma^a \left(\partial_\mu + \omega_{\mu ab}(e)\, \gamma^{ab}\right), \tag{13}$$

where e_a^μ is orthogonal frame. $\omega_{\mu ab}(e)$ is Riemannian (spinor) connection corresponding to e_a^μ .

Then axial current anomaly is given [1] by the formula

$$\nabla_\mu J^{\mu 5} = \frac{2}{(2\pi)^{d/2}} \int d\psi^0 \prod_{i=1}^{d/2} \frac{\lambda_i}{sh\, \lambda_i}\ , \tag{14}$$

where λ_i are eigenvalues of the matrix $\hat{R}_{ab} = R_{cdab} \varphi_c^{\,0} \varphi_d^{\,0}$.
R_{cdab} is the curvature of the connection $\omega_{\mu ab}(e)$.

Let us consider a more general operator $\mathcal{D}(\omega)$ with connection $\omega_{\mu ab} = \omega_{\mu ab}(e) + S_{\mu ab}$, $S_{\mu ab}$ is the third-rank tensor:

$$\mathcal{D}(\omega) = i e^{\mu}_{a} \gamma^{a} \left(\partial_\mu + \omega_{\mu ab} \gamma^{ab} \right). \tag{15}$$

The operator (15) reduces to our operator $\mathcal{D} = i\gamma^\mu(\partial_\mu - A_{\mu\nu\lambda}\gamma^{\nu\lambda})$ if $e^{\mu}_{a} = \delta^{\mu}_{a}$, $\omega(e) = 0$, $S_{\mu ab} = A_{\mu ab}$, and reduces to (13) if $S_{\mu ab} = 0$.

It is natural to suspect that Eq.(14) must be valid for (15) too, if λ_i in (14) are taken to be eigenvalues of the matrix $\hat{R}_{ab} = R_{cdab} \varphi_c^{\,0} \varphi_d^{\,0}$, where R_{cdab} is now the curvature of the connection $\omega = \omega(e) + S$. But our result (12) shows that for the particular case $e^{\mu}_{a} = \delta^{\mu}_{a}$, $S_{\mu ab} = A_{\mu ab}$ it does not take place - the curvature in (12) is the one defined by the connection $3A_{\mu ab}$ rather than by $A_{\mu ab}$, i.e. the anomaly for the operator (15) is given probably by Eq.(14), where λ_i are defined by the curvature of the connection $\omega_{\mu ab}(e) + 3S_{\mu ab}$ rather than $\omega_{\mu ab} = \omega_{\mu ab}(e) + S_{\mu ab}$.

It is worthwhile noting the following.

As is well known, the integral of the anomaly (see, e.g. [1]) $\int dx\, \partial_\mu J^5_\mu$ for compact manifolds is the integer number - the index of operator $\mathcal{D}$ - and is equal to the number of the left-hand zero modes of the operator $\mathcal{D}$ minus the number of the right-hand zero modes. (It is readily seen from the equation $\int dx\, \partial_\mu J^5_\mu = Sp\, \gamma^5 e^{-t\mathcal{D}^2}$). According to the Atiyah-Singer theorem (see, e.g. [20]) the index of the operator (15) is equal to the integral of the r.h.s. of Eq.(14), where λ_i are eigenvalues of the curvature defined by some connection of the manifold, in particular the connections that differ by the torsion tensor give the same result for the index (it is obvious because the torsion $S_{\mu ab}$ can be changed continuously, in particular it may turn to zero, whereas the index is integer and doesn't change).

Thus the integral of the anomaly is given by the Atiyah-Singer theorem and coincides with the results (12) and (14).

The appearance of the curvature tensor corresponding to the connection $3A_{\mu\nu\lambda}$ may be seen directly when considering $Sp\gamma^5 exp(-\frac{t}{2}\not{D}^2)$. Namely, calculating $\not{D}^2$ we obtain

$$\varphi(t,x) = tr\gamma^5 exp(-\frac{t}{2}\not{D}^2) =$$

$$= tr\gamma^5 exp\left\{\frac{t}{2}\left[(\partial_\mu + A_{\mu\nu\lambda}3\gamma^{\nu\lambda})^2 + F_{\mu\nu\lambda\rho}\gamma^{\mu\nu\lambda\rho} + 12A_{\mu\nu\lambda}A_{\mu\nu\lambda}\right]\right\}$$

If $F_{\mu\nu\lambda\rho} \neq 0$, then the expansion of $\varphi(t,x)$ at $t \to 0$ begins from the negative powers of t (as was pointed out in Sect.1). For example, if d = 4, the straightforward calculation of the Seeley coefficients $\varphi_{-4}, \dots, \varphi_0$ gives

$$\varphi(t,x) = \frac{1}{2\pi^2 t}\partial_\mu B_\mu + \frac{1}{24\pi^2}\partial_\mu\left(B_\alpha \partial_\nu B_\beta \varepsilon^{\mu\alpha\nu\beta} + \partial^2 B_\mu + 4B_\mu B^2\right) + \dots$$

$$B_\mu = \varepsilon_{\mu\nu\alpha\beta}A_{\nu\alpha\beta} \, .$$

If the condition (11) $F_{\mu\nu\lambda\rho} = 0$ is applied (as in Sect.1), then

$$\varphi(t,x) = tr\gamma^5 exp\left\{\frac{t}{2}\left[(\partial_\mu + A_{\mu\nu\lambda}3\gamma^{\mu\nu\lambda})^2 + A_{\mu\nu\lambda}A_{\mu\nu\lambda}12\right]\right\}. \quad (16)$$

The anomaly is given by the coefficient O(1) in $\varphi(t,x)$ expansion at $t \to 0$.

It is evident that gauge invariance arises in expression (16) if we omit the term $6t A_{\mu\nu\lambda}A_{\mu\nu\lambda}$ in the exponent. The field $3A_{\mu\nu\lambda}\gamma^{\nu\lambda}$ transforms as the SO(d) gauge field in the spinor representation. So in this case it is obvious that $\varphi(t,x)$ begins from the order O(1).

The omitted term in (16) is of order t and does not contain γ^{μ} - matrix, so it can change only the terms of order t and higher in $\varphi(t,x)$ and its contribution to anomaly is zero, as it was mentioned above in the other terms.

It follows from the above qualitative considerations that the result for anomaly is expressed through the strength of the connection $3A_{\mu\nu\lambda}\gamma^{\nu\lambda}$, i.e. through

$$\left[\partial_\mu + 3A_{\mu\alpha\lambda}\gamma^{\alpha\lambda}, \partial_\nu + 3A_{\nu\sigma\rho}\gamma^{\sigma\rho}\right] = \left(3\partial_\mu A_{\nu\lambda\rho} - 3\partial_\nu A_{\mu\lambda\rho} + 72 A_{\mu\lambda\alpha}A_{\nu\alpha\rho}\right)\gamma^{\lambda\rho}$$

which coincides with tensor $R_{\mu\nu\lambda\rho}$ introduced earlier.

We want to note now that difference between the Lagrangian in the functional integral (4) and the classical one (5) is proportional to the scalar curvature of this curvature tensor. Indeed,

$$\delta^{\mu\lambda}\delta^{\nu\rho}R_{\mu\nu\lambda\rho} = 72\, A_{\mu\nu\lambda}A_{\mu\nu\lambda}\,,$$

The similar addition to the classical Lagrangian - the scalar curvature of the curvature tensor defined by the Riemannian connection - has been found in [18] in the case of scalar particle propagating in the external gravitational field.

Conclusion.

In this work we have studied the external tensor fields contribution to anomalous divergence of the Dirac fermions axial current. We have obtained the supersymmetric models the supercharge of which after quantization transforms to the Dirac operator in the external tensor fields of arbitrary odd rank. It is shown that Gaussian approximation for path integral representation of Witten's partition function of these models gives the anomaly only in the case of the third-rank external tensor field with vanishing external derivative.

In this case we calculate anomaly in arbitrary dimension. In other cases the answer will be given by the definite order of the perturbation theory around the Gaussian approximation.

For the third-rank tensor fields with vanishing external derivative we also show that the action in the path integral differs from the classical one by the extra term that is analogous to the term $R/6$ that was found by De' Witt in the other situation - scalar particle in the external gravitational field.

This extra term may be interpreted also as scalar curvature, this time, of the curvature tensor defined by non-Riemannian connection.

The Dirac operator that is under consideration could be interpreted as the one in the external gravitational field with torsion. With this interpretation our results are compared to those of [1] and to Atiyah-Singer index theorem.

The results obtained here may be applied to the supergravity theories in which tensor fields, in particular the third-rank tensor fields with vanishing external derivatives belong to the fields supermultiplet. Especially the calculation of the external tensor field contribution to the anomalous divergence of the spin 1/2 and 3/2 fields energy-momentum tensors is of great interest. According to [1], this divergence in dimension d is directly connected with the axial anomaly in the dimension $d + 2$.

We are indebted to A.A.Tseytlin and A.S.Schwarz for valuable discussions.

APPENDIX

Symbols of the Operators.

Let we have operator $\hat{A}$ that acts in the space of the d-dimensional Dirac spinors. In other words, it is a matrix of $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ dimensionality. We assume d to be even. We expand $\hat{A}$ with respect to full set of anti-symmetrized products of Dirac matrices:

$$\hat{A} = C_0 + \frac{1}{\sqrt{2}} C_\mu \gamma^\mu + \frac{1}{2} C_{\mu\nu} \gamma^{\mu\nu} + \ldots + \frac{1}{2^{\frac{d}{2}}} C_{\mu_1 \ldots \mu_d} \gamma^{\mu_1 \ldots \mu_d} \tag{A.1}$$

The Weyl symbol of operator $\hat{A}$ is the following function of Grassmannian variables ψ^μ

$$Symb\, \hat{A} = A(\psi) = C_0 + C_\mu \psi^\mu + C_{\mu\nu} \psi^\mu \psi^\nu + \ldots \tag{A.2}$$

Evidently, the operator is received uniquely from its symbol:

$$\hat{A} = \frac{1}{2^{d/2}} \int d\psi \, d\psi' A(\psi)\, e^{i(\gamma^\mu - \sqrt{2}\,\psi^\mu)\psi'^\mu} \tag{A.3}$$

Eqs.(A.1)-(A.3) set up one-to-one correspondence between the symbols (the functions of ψ^μ) and the operators.

Let $\hat{A}$ and $\hat{B}$ be matrices, A and B their symbols. Then the symbol $Symb(\hat{A}\hat{B})$ of $\hat{A}\hat{B}$ is expressed through the symbols $\hat{A}$ and $\hat{B}$ by the formula

$$Symb\, \hat{A}\hat{B} = (A * B)(\psi) = \tag{A.4}$$

$$= \frac{1}{2^d} \int d\psi_1 \, d\psi_2 \, A(\psi_1) B(\psi_2) \, e^{2(\psi\psi_1 + \psi_1\psi_2 + \psi_2\psi)} ,$$

The analogous formulae hold also in the bosonic case if we consider vectors $z_m = (p_\mu, x_\mu)$ in 2d-dimensional phase space ($m = 1, 2, \ldots 2d$; instead of ψ^μ (x_μ are coordinates, p_μ are momenta). Matrices γ^μ are substituted by operators $\hat{z}_m = (\hat{p}_\mu, \hat{x}_\mu)$. The number of terms in expansion (A.1) of operator $\hat{A}$ acting now in the space of the scalar functions of x_μ is generally infinite. To be more exact, the operator $\hat{A}$ must be expanded now in the symmetrized products of the operators :

$$\hat{A} = c_0 + c_m \hat{z}_m + c_{mn} \hat{z}_m \hat{z}_n + \ldots$$

Then the function

$$A(z) = c_0 + c_m z_m + c_{mn} z_m z_n + \ldots$$

is the Weyl symbol of the operator $\hat{A}$.

More formally, we say that the symbol of the operator

$$\hat{A} = \int d\alpha \, \varphi(\alpha) \, e^{i(\alpha \hat{z})} , \quad (\alpha \hat{z}) = \alpha_1 \hat{z}_1 + \ldots + \alpha_{2d} \hat{z}_{2d}$$

is the function

$$A(z) = \int d\alpha \, \varphi(\alpha) \, exp\left(i(\alpha z)\right)$$

The following formula is analogous to (A.4) for the bosonic case

$$\mathrm{Symb} \, \hat{A}\hat{B} \, (z) = (A * B)(z) =$$

$$= \frac{1}{\pi^{2d}} \int dz^1 dz^2 \, A(z^1) B(z^2) \, e^{2i(z\omega z^1 + z^1 \omega z^2 + z^2 \omega z)},$$

where $\quad z\omega z' = p_\mu x'_\mu - p'_\mu x_\mu .$

For details see Refs [16,17] .

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The manuscript was received 31 October 1985

Д.А.ЗУРАБЯН, Р.Л.МКРТЧЯН, О.М.ХУДАВЕРДЯН

ОБ АКСИАЛЬНЫХ АНОМАЛИЯХ ВО ВНЕШНИХ ТЕНЗОРНЫХ ПОЛЯХ

(на английском языке, перевод З.Н.Асланян)

Редактор Л.П.Мукаян

Технический редактор А.С.Абрамян

Подписано в печать 30/XII-85г. ВФ-09259 Формат 60x84/16
Офсетная печать.Уч.изд.л. I,0 Тираж 299 экз. Ц. I5 к.
Зак.тип. № 632 Индекс 3624

Отпечатано в Ереванском физическом институте
Ереван 36, Маркаряна 2

индекс 3624

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ