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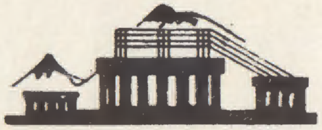

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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STELLAR SYSTEMS AS DISSIPATIVE DYNAMICAL SYSTEMS

ЦНИИатоминформ

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ԱՍՏՂԱՅԻՆ ՀԱՄԱԿԱՐԳԵՐԸ ՈՐՊԵՍ ԴԻՍՍԻՊԱՏԻՎ ԴԻՆԱՄԻԿԱԿԱՆ ՀԱՄԱԿԱՐԳԵՐ

Ուսումնասիրված է կենտրոնական զանգված պարունակող աստղային համակարգերի էվոլյուցիան, դինամիկական համակարգերի և արհավիրքների տեսությունների օգնությամբ: Խնդիրը հանգեցվում է երկչափ դիսսիպատիվ համակարգերի՝ պարզ ատրակտորի հետազոտմանը: Ցույց է տրված կայուն և անկայուն հատուկ կետերի /հանգույցների, կիզակետերի/, սահմանային պարբերաշրջանների առկայությունը: Համակարգում կարող են տեղի ունենալ ենթակրիտիկական և վերկրիտիկական Հոպֆի ճյուղավորումներ /բիֆուրկացիաներ/, որոնց տարաբաժան գծերը /սեպարատրիսները/ համապատասխանում են $A_{\pm 5}$ 'թիթեռ' տիպի համաչափ արհավիրքին:

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1. Statement of the Problem.

The investigation of the dynamics of the stellar systems containing massive central body has become of particular urgency during the last decade. This is conditioned, on the one hand by some recently received unusual observational data (X-ray sources in globular clusters, anomalous brightness spike in the centre of some clusters and galaxies, etc.), on the other hand by the development of theoretical representations on formation and evolution of massive black holes in the Universe. Beginning with the pioneer works of Lynden-Bell, Peebles, Hills and others, a great progress is achieved in this field; e.g. Shapiro [16] had noticed that a massive central object can essentially affect the evolution of the system in certain cases even halting its collapse.

In the present paper, an attempt is made to study the evolution of such a system from the viewpoint of the theory of dynamical systems. As distinct from Refs [1-4], where by similar methods the stellar systems as Hamiltonian dynamical systems were studied. here we shall consider open non-conservative systems for which the Liouville theorem on phase volume conservation is not fulfilled.

While in Ref. [5] we dealt with the stability of this system, our present aim consists in studying the evolutionary equations, i.e. those determining time variation for the basic physical parameters of the stable system. Naturally, these equations will be derived in certain simplifications and generalizing assumptions. We believe that some rather general features of real systems can be revealed here.

We consider a spherically symmetric system with nonconserved total number of particles (stars). Here we take into account the following main processes leading to a decrease of this number: evaporation of high-energy stars from the system [6] and capture of stars by the massive object, more precisely, their tidal disruption in the Roche lobe [7] . Simultaneously, for completeness, we involved in our consideration possible processes of the system replenishment with stars. A suppression of the stochastic effects in the massive body dynamics is assumed [8] .

During the analysis of the problem we essentially use the methods developed in the theory of dynamical systems, in particular, of dissipative systems, as well as in the catastrophe theory. The problem is reduced to the study of a two-dimensional dissipative system which turned out rather rich in its diverse manifestations as a simple attractor. Thus, the presence of stable and unstable singular points (nodes, focuses) as well as stable and unstable limit cycles is shown. Subcritical and supercritical Hopf bifurcations with a separatrix corresponding to a symmetrical $A_{\pm 5}$ -"butterfly" type catastrophe (by Thom's classification) can occur.

Already the results of our, of course, simplified analysis point out the large diversity in paths of evolution of the considered systems versus the prescribed physical parameters.

2. Derivation of Basic Equations.

Let we have a spherically symmetric system of N gravitating particles of equal mass m with a massive point-like centre of a mass $M \gg m$. So long as the evolutionary time scales conditioned by evaporation and star capture exceed much the relaxation time scale of the systems we are interested in, the virial theorem can be applied to the latters. Then the system's total energy can be determined from the relation [6]

$$E = -\frac{1}{4}\,\frac{G N^2 m^2}{R}\,, \tag{2.1}$$

where R is the system's characteristic radius.

The rate of variation of total number of stars of the system can be presented as follows:

$$\dot{N} = \dot{N}_{ev} - F + d\,, \tag{2.2}$$

where

$$\dot{N}_{ev} = -a N \tag{2.3}$$

is the stellar evaporation rate [6] (a is the part of stars leaving the system during the relaxation time). The flux of stars into the Roche lobe of the massive object (the contribution of finite stars being neglected) is [7] :

$$F = b R^{-1}. \tag{2.4}$$

It is known also that not all stars from the flux (2.4) will be destroyed, i.e. vanish from the system; a certain part of perturbed stars can return to the system (see, e.g. [9,10]). This effect of stars' arrival back is determined mainly by the kinetics of tidal disruption, i.e. by the internal structure of stars rather than by the system parameters. The contribution of possible star-formation processes can also be subscribed to the last term in (2.2).

Differentiating (2.1) with respect to time we arrive at the following equation

$$\frac{\dot{R}}{R} = \frac{\dot{N}}{N}\left(2 - \frac{\dot{E}}{E}\frac{N}{\dot{N}}\right), \qquad (2.5)$$

in which the energy flux due to the stars' loss $\dot{E}$ can be presented in the form (for details see [16,11]):

$$\frac{\dot{E}}{E} = -\frac{kb}{N^2} + c, \qquad (2.6)$$

where the constant c corresponds to term d in (2.2).

Hence the problem is reduced to investigation of a system of two nonlinear equations determining variation of system's basic parameters - the radius and total number of stars:

$$\dot{N} = -aN - \frac{b}{R} + d,$$
$$\frac{\dot{R}}{R} = -2a - \frac{2b}{RN} + \frac{2d}{N} + \frac{kb}{N^2} - c. \qquad (2.7)$$

3. Two-Dimensional Dissipative Systems.

First, let us summarize some notions from the theory of dynamical system and catastrophe theory to be used in the further analysis. The detailed description of these concepts can be found in the books [12-15] .

Consider a two-dimensional system described by the following differential equations:

$$\frac{dx}{dt} = P(x,y),$$
$$\frac{dy}{dt} = Q(x,y), \qquad (3.1)$$

where $P(x,y)$ and $Q(x,y)$ are smooth functions.

According to the Liouville theorem, the element of phase volume of the Hamiltonian system is conserved. One can show that in general the variation rate of the elementary volume $\Delta\tau(x,y)$ at the point (x, y) is

$$\Lambda(x,y) = \frac{1}{\Delta\tau(x,y)}\frac{d}{dt}\left[\Delta\tau(x,y)\right] = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The system is called dissipative if $\Lambda(x,y)$ is not equal to zero identically.

Assume that there exists a region Ω whence no trajectories of the system (3.1) come out and where $\Lambda(x,y) < 0$. Then stable stationary motion on Ω must take place in a region of lesser dimension. Such manifolds are called attractors.

For a limited two-dimensional flux, according to the Poincare-Bendickson theorem, only two types of attractors are possible:

1. stable immobile points (stable focuses);
2. limit cycles.

The Dulak-Bendickson criterion points out when the system has no limit cycles. If in some region M there exists such a smooth $F(x,y) \neq 0$ function that

$$\frac{\partial(FP)}{\partial x} + \frac{\partial(FQ)}{\partial y} > 0, \qquad (3.2)$$

then the system in this region has no limit cycles. Indeed, on limit cycle

$$\oint_{\ell} P\,dy - Q\,dx = 0,$$

while by the Stockes theorem and from (3.2)

$$0 = \oint\limits_{l} F P dy - F Q dx = \int\limits_{S} \left[\frac{\partial (FP)}{\partial x} + \frac{\partial (FQ)}{\partial y} \right] dx\, dy > 0,$$

hence the system has no limit cycles on M and can have only attractors (simple) of the 1-st type, namely stable singular points.

The basic point in the qualitative investigation of dynamical system (3.1) is the study of singular points, equilibrium positions or rest points (x_0, y_0) on which $P(x_0,y_0) = Q(x_0,y_0) = 0$. Here important characteristics are the eigenvalues determined from the linearized equations:

$$\delta\dot{x} = \frac{\partial P}{\partial x}\delta x + \frac{\partial P}{\partial y}\delta y, \qquad (3.3)$$

$$\delta\dot{y} = \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta x,$$

or

$$\frac{d}{dt}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = A\begin{pmatrix}\delta x \\ \delta y\end{pmatrix},$$

where

$$A = \begin{Vmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{Vmatrix}.$$

Eigenvalues of matrix A are defined by the following equation:

$$\det \begin{Vmatrix} \dfrac{\partial P}{\partial x} - \zeta & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} - \zeta \end{Vmatrix} = 0. \qquad (3.4)$$

Eigenvalues $\zeta_{1,2}$ do not vary at any regular (at (x_0, y_0) point) excnange of coordinates x, y. The singular point (x_0, y_0) is called non-degenerate if $\mathrm{Re}\,\zeta_i \neq 0$, $i = 1,2$. The behaviour of trajectories of the system (3.1) in a small vicinity of non-degenerate point is qualitatively equivalent to the behaviour of trajectories of its linear part (3.3).

Let us briefly give the classification of the singular points. If the eigenvalues ζ_1, ζ_2 are real and of the same sign, then the singular point is a node (Fig.1a). The non-degenerate node ($\zeta_1 \neq \zeta_2$) is called attractive (stable) if $\zeta_1, \zeta_2 < 0$. The node is called repulsive (unstable) if $\zeta_1, \zeta_2 > 0$: the relevant phase picture is obtained from Fig.1a by reversal of time. At $\zeta_1 = \zeta_2 = \zeta$ one has a degenerate node. At $\zeta > 0$ the node is unstable, at $\zeta < 0$ - stable (Fig.1b).

In case the eigenvalues ζ_1, ζ_2 are complex-conjugated, we have a focus. At $\mathrm{Re}\,\zeta_i < 0$ this focus is attractive. The trajectories rotation direction is determined by the sign $\mathrm{Im}\,\zeta_1 > 0$ (Fig.1a). At $\mathrm{Re}\,\zeta_i > 0$ the focus is repulsive with a direction corresponding to the time inversion.

When the eigenvalues of some singular point are purely imaginary: $\mathrm{Re}\,\zeta_i = 0$, then it is a vortex (Fig.1d). The vortex is structurally unstable, therefore a qualitative behaviour of the dynamical system in its vicinity depends essentially on nonlinear terms. The vortex's distortion is convenient to consider in polar coordinates. Then the equations of motion will have the form

$$\frac{dr}{dt} = \lambda r, \qquad (3.5)$$

$$\frac{d\vartheta}{dt} = \omega \neq 0.$$

As λ passes zero, a change of dynamical stability takes place. A quite general distortion of dynamical system (3.5) has the form:

$$\frac{dr}{dt} = \lambda r + Ar^3 + Br^5 + \dots$$

$$\frac{d\vartheta}{dt} = \omega . \qquad (3.6)$$

For one-parametrical family of dynamical systems the (3.6) can be re-written in the form:

$$\frac{dr}{dt} = \lambda r \pm r^3 ,$$

$$\frac{d\vartheta}{dt} = \omega . \qquad (3.7)$$

In such a system the dependence $\vartheta(t)$ is trivial: $\vartheta(t) = \vartheta_0 + \omega t$.

Consider now the stationary values of r for the given dynamical system (3.7) in the case

$$\frac{dr}{dt} = \lambda r - r^3$$

or

$$\frac{dr}{dt} = - \frac{d}{dr} V(r, \lambda) ,$$

where

$$V(r, \lambda) = - \frac{\lambda r^2}{2} + \frac{r^4}{4} . \qquad (3.8)$$

A qualitative behaviour of the families of functions depending on several parameters is studied in the catastrophe theory. In this theory a potential of the (3.8) form is called an $A_{\pm 3}$ catastrophe (symmetrical) of a "cusp" type. In the parameters' space (in the given (3.7), (3.8) cases it is a line) there exist surfaces (lines, points) of lesser dimension, called separatrices, which divide the space into separate regions. Functions in one and the same region have similar qualitative behaviour (equal number of minima, maxima). On the separatrices the functions have non-Morse critical points, i.e. $V'(x) = V''(x) = 0$.

Let us determine the separatrix of the families

$$\frac{d}{dr} V(r, \lambda) = - \lambda r + r^3 = 0 ,$$

$$\frac{d^2}{dr^2} V(r, \lambda) = - \lambda + 3r^2 = 0 . \qquad (3.9)$$

The system (3.9) has a unique solution $\lambda = 0$, $r = 0$. The separatrix of the parameter space λ consists of a point $\lambda = 0$. One can readily see that at $\lambda < 0$ the function $V(r, \lambda)$ has one minimum, while at $\lambda > 0$ two minima and one maximum (Fig.2a).

For the system (3.7), (3.8) this means that at $\lambda < 0$ there exists a stable point ($r = 0$), while at $\lambda > 0$ this point is unstable and a stable limit cycle occurs. This implies that as λ passes through zero the stable focus becomes unstable forming around it a stable limit cycle with a radius $r = \sqrt{\lambda}$. This phenomenon is known as supercritical Hopf bifurcation.

For the system (3.7) at

$$V(r, \lambda) = - \frac{\lambda r^2}{2} - \frac{r^4}{4}$$

the point $r = 0$ is stable at $\lambda < 0$ and unstable at $\lambda > 0$. When $\lambda < 0$, there exists an unstable limit cycle with a radius $r = \sqrt{-\lambda}$ (Fig.2b), and one has a subcritical Hopf bifurcation (or inverse Hopf bifurcation).

When a dynamical system depends on two independent parameters, the (3.6) in the general case has a form:

$$\frac{dr}{dt} = \lambda r + 2\mu r^3 \pm r^5, \qquad \frac{d\vartheta}{dt} = \omega. \tag{3.10}$$

Analogously we shall find separatrices in (λ, μ) plane. It can be easily shown that the separatrices have the form shown in Fig.3a at $V(r;\lambda,\mu) = -\frac{\lambda r^2}{2} - \frac{\mu r^4}{2} + \frac{r^6}{6}$. According to Thom's classification, this catastrophe is symmetrical of $A_{\pm 5}$ "butterfly" type. In this case the bifurcation depends on the path in the parameters space.

Let us consider three typical paths 1, 2, 3 (Fig.3b). Obviously, path 1 corresponds to supercritical Hopf bifurcation. On path 2 λ remains constant and one stable and one unstable cycles are formed. On path 3 just like on path 2 two cycles are formed, but the inner cycle vanishes while traversing through the separatrix.

In case when

$$V(r;\lambda,\mu) = -\frac{\lambda r^2}{2} - \frac{\mu r^4}{2} - \frac{r^6}{6}$$

a qualitatively analogous picture arises. only the stable points (cycles) turn into unstable ones and vice versa.

4. Qualitative Analysis of Dissipative Stellar System.

Let us rewrite the obtained in Sect. 2 system of equations (2.7) in the following form:

$$\dot N = -a_1 N - \frac{a_2}{R} + a_3, \qquad \dot R = -(2a_1 + a_5)R - \frac{2a_2}{N} + \frac{2a_3}{N}R + \frac{a_4}{N^2}R, \tag{4.1}$$

where $a_i > 0$, $i = 1,\ldots,5$ and $R > 0$. $N > 0$.

Making in (4.1) exchange of variables

$$N = \frac{a_3}{a_1} e^x, \quad R = \frac{a_2}{a_3} e^{y+2x}, \quad t = a_1 t_*, \tag{4.2}$$

we shall have

$$\dot x = -1 - e^{-3x-y} + e^{-x}, \qquad \dot y = b_1 e^{-2x} - b_2, \tag{4.3}$$

where

$$b_1 = \frac{a_1 a_4}{a_3^2}, \quad b_2 = \frac{a_5}{a_1}.$$

Thus we have obtained a system of equations depending on two parameters b_1, b_2. Note that the initial system (4.1) depends on five parameters: as follows from (4.2), three parameters are used in normalization of N, R and t.

One can readily see that the system (4.3) has a unique singular point. From $\dot x = \dot y = 0$ conditions we obtain

$$x_0 = \frac{1}{2} \ln \frac{b_1}{b_2}, \qquad y_0 = -\ln\left[\frac{b_1}{b_2}\left(1 - \sqrt{\frac{b_1}{b_2}}\right)\right], \tag{4.4}$$

provided that $1 - \sqrt{\frac{b_1}{b_2}} > 0$. We shall deal with these parameters only.

Consider now the system's behaviour near the singular point (4.4). Linearizing Eqs (4.3) in the vicinity of the point (x_0, y_0) and inserting notations

$$q = \sqrt{\frac{b_2}{b_1}} - \frac{3}{2} > -\frac{1}{2},$$

$$p = 2b_2\left(q + \frac{1}{2}\right),$$

we obtain

$$\delta\dot{x} = 2q\,\delta x + \left(q + \frac{1}{2}\right)\delta y,$$

$$\delta\dot{y} = -\frac{p}{q + \frac{1}{2}}\,\delta x \qquad (4.5)$$

or

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = A\begin{pmatrix}\delta x\\ \delta y\end{pmatrix},$$

where

$$A \doteq \begin{Vmatrix} 2q & q + \frac{1}{2} \\ -\frac{p}{q + \frac{1}{2}} & 0 \end{Vmatrix}.$$

Eigenvalues of A are:

$$\zeta_{1,2} = q \pm \sqrt{q^2 - p} = q \pm \sqrt{D}. \qquad (4.6)$$

The phase pictures versus parameters q and p have the form shown in Fig.4.

One can see from Fig.4 that at $q > 0$ the singular point is unstable, at $D > 0$ it is a repulsive node, at $D = 0$ a degenerate repulsive node, at $D < 0$ a repulsive focus. At $q < 0$ the same phase pictures take place, only in this case the singular point is attractive.

Here an important question arises whether the given system has limit cycles.

One can see from (4.1) that

$$\Lambda(N, R) = \frac{\partial\dot{N}}{\partial N} + \frac{\partial\dot{R}}{\partial R} = -3a_1 - a_5 + \frac{2a_3}{N} + \frac{a_4}{N^2}.$$

If N_1 is a positive solution of equation $\Lambda(N, R) = 0$ (it is easy to be convinced that there always exists a unique positive solution), then $\Lambda > 0$ at $N < N_1$, while $\Lambda < 0$ at $N > N_1$. Then from the cited in Sect. 3 Dulak-Bendickson criterion, it follows that no limit cycles exist in the regions $N < N_1\ (F = 1)$, $N > N_1\ (F = -1)$. Hence, if the limit cycles really exist, they must intersect the line $N = N_1$.

Let us now see what will happen with the system near the singular point when parameter q goes through zero. So far as the system (4.3) depends on two parameters, then according to Sect. 3, near the singular point the system can be transformed into the system (3.10). From the analysis of system (3.10), i.e. the symmetrical $A_{\pm\bar{5}}$ "butterfly" type catastrophe it follows that for the system (3.1) there exist parameters at which it has no limit cycles, has a single limit cycle (stable if the singular point is unstable (Fig.5a) and vice versa (Fig.5b)) or has two limit cycles (Fig.5c,d).

5. Conclusion.

The present paper deals with the evolution of a spherically symmetrical stellar system containing a central massive body. The problem is reduced to the analysis of two-dimensional nonconservative dynamical system. From the theory of dissipative dynamical systems, particularly from the Poincare-Bendickson theorem follows the impossibility of chaotic motion for two-dimensional systems. Hence the problem is reduced to the study of a simple attractor, i.e. to the problem on the existence and stability of singular points, limit cycles as well as bifurcations. It turned out that versus the

the free parameters the system may have a structurally stable unique singular point (4.4) which may be either dynamically stable or unstable including repulsive (attractive) node and focus. It is shown that versus the parameters the system (2.7) may possess two limit cycles, one limit cycle or none of them. In particular, versus the path (see Fig.4) in space, a supercritical and subcritical Hopf bifurcations can be realized. The resulting separatrix corresponds to the symmetrical catastrophe of $A_{\pm 5}$ "butterfly" type.

The reduction of the problem to the two-dimensional dynamical system was justified by the fact that as mentioned in Sect. 2, the characteristic evolutionary time scales exceed much the relaxation time scale of the real systems. Analogously, the variation of the system's other parameters, such as, e.g. the central mass, mean stellar concentration entering the expression for the star tidal disruption rate ($\dot{N} \propto M^{4/3} n^{1/2} R^{-1}$), according to the results of a number of investigations (see, e.g. [11] and the papers cited therein), is extremely weak in a wide range of parameters as compared to variation of radius R or star number N .

The results of our analysis demonstrate the power of dynamical methods for determination of evolutionary paths of the stellar systems versus the physical parameters. The detailed studies are surely needed in future.

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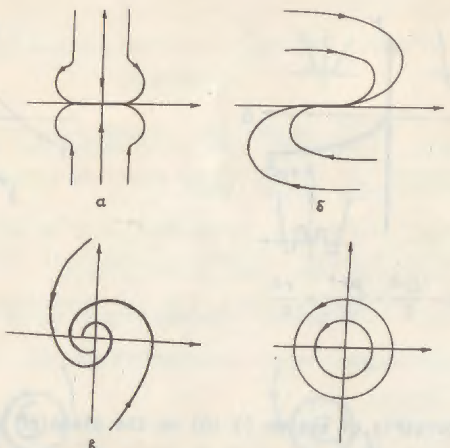

Fig.1. Classification of singular points: a) node; b) degenerate node; c) focus; d) vortex.

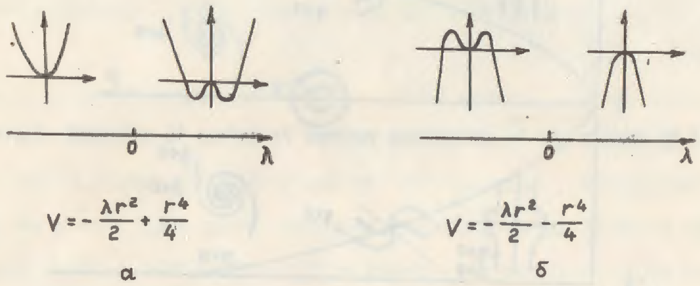

Fig.2. Behaviour of function $V(r, \lambda)$ and the separatrix $\lambda = 0$.

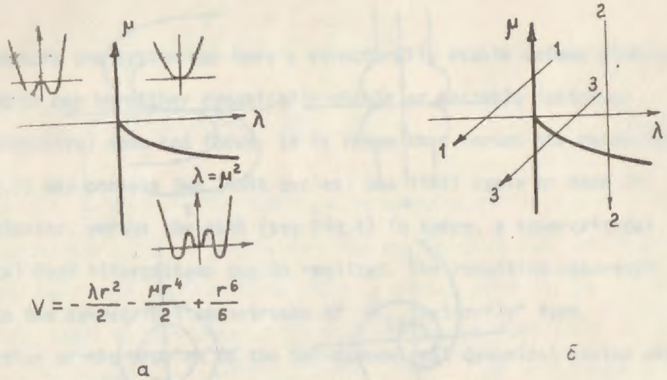

Fig.3. Separatrix of system (3.10) on the plane of parameters λ and μ .

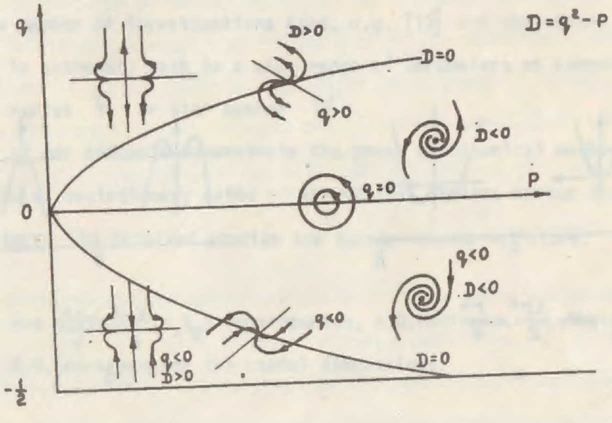

Fig.4. Phase picture of system (3.1) and character of singular points.

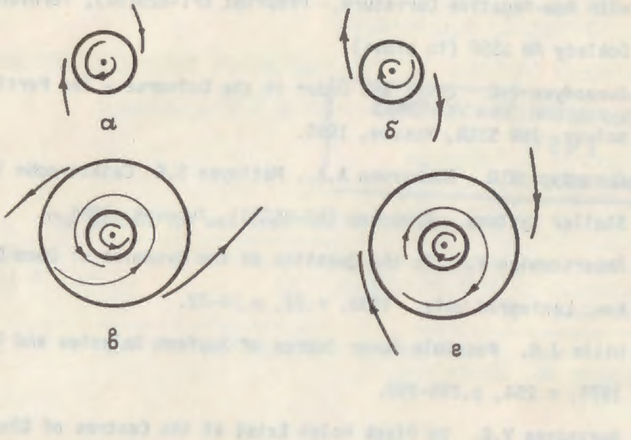

Fig.5. Presence of cycles at various parameters of the system (3.1).

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