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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

A.Z. AKHEYAN, N.S. ANANIKYAN

CRITICAL BEHAVIOUR OF THE INFINITE-DIMENSIONAL POTTS MODEL

ЦНИИатоминформ

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ՓՈԹԻ ՄՈԴԵԼԻ ԿՐԻՏԻԿԱԿԱՆ ՀԱՏԿՈՒԹՅՈՒՆՆԵՐԸ ԲԵՏԵԻ ՑԱՆՑԻ ՎՐԱ

Ուսումնասիրվել է Փոթի Q-բաղադրիչային մոդելի կրիտիկական վարքը Բեթեի ցանցի վրա: Գտնվել են 2-րդ կարգի փուլային անցման $T_c^{(1)}$ կրիտիկական և սպոնտան մագնիսացվածության առաջացման $T_c^{(2)}$ ջերմաստիճանները, որոնք համընկնում են $Q = 2$ Իզինգի մոդելի համար: Հաշվվել են β և δ կրիտիկական ցուցիչները ինչպես նաև սկեյլինգի ֆունկցիան $T \sim T_c^{(1)}$ համար: Ստացված կրիտիկական ցուցիչները համապատասխանում են միջին դաշտի արդյունքները Իզինգի մոդելի համար, և հավասար են $\beta=\frac{1}{2}$, $\delta=3$: Նրանց արժեքը կախված չէ Q-ից: Դուրս է գրվել նաև մոդելի ազատ էներգիայի արտահայտությունը:

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Н.С.АНАНИКЯН, А.З.АХЕЯН

КРИТИЧЕСКИЕ СВОЙСТВА МОДЕЛИ ПОТТСА НА РЕШЕТКЕ БЕТЕ

Изучено критическое поведение Q - компонентной модели Поттса на решетке Бете. Найдены критические температуры $T_c^{(1)}$ - фазового перехода II рода и $T_c^{(2)}$ - возникновения спонтанной намагниченности. Эти температуры совпадают для модели Изинга $Q = 2$. Вычислены критические индексы β и δ, а также функция скейлинга при $T \sim T_c^{(1)}$. Полученные критические индексы подтверждают результаты среднего поля для модели Изинга и равны $\beta = \frac{1}{2}$, $\delta = 3$. Значение их не зависит от Q. Выписано также выражение свободной энергии модели.

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A.Z. AKHEYAN, N.S. ANANIKYAN

CRITICAL BEHAVIOUR OF THE INFINITE-DIMENSIONAL POTTS MODEL

The critical behaviour of the Q-component Potts model on a Bethe lattice is studied. Critical temperatures $T_c^{(1)}$ of the II order phase transition and $T_c^{(2)}$, below which a spontaneous magnetization exists, are found. These temperatures coincide for the Ising model $Q = 2$. Critical exponents β and δ, and also scaling function at $T \sim T_c^{(1)}$ are calculated. The obtained critical exponents confirm the mean field results for the Ising model and are $\beta = \frac{1}{2}$, $\delta = 3$, respectively. Their values are independent of Q.

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For the spin Potts model the topology is nontrivial in any dimensions of $d > 1$, and is due to the contribution of the loops - ways with nontrivial first homology group [1]. However, the loops disappear in another critical case, i.e. when the Hausdorff dimension of the lattice $d_H = \infty$. It is realized on a Cayley tree, and one may expect that the Potts model, formulated on that lattice, can be exactly solved.

The present paper is devoted to the study of critical properties of the Potts model on a Bethe lattice at an arbitrary external field.

The Cayley tree, on which the model will be formulated, is shown in Fig.1. This graph is characterized by the coordination number q (the number of the nearest neighbours of each site). The number of sites of the r-th shell of the graph (sites - r steps distant from the central point) is $q(q-1)^{r-1}$, and the number of sites in n shells is altogether $C_n = q[(q-1)^n - 1]/(q-2)$. At $n \to \infty$ the Hausdorff dimension, determined by the ratio

$$d_H = \lim_{n\to\infty} \frac{\ln C_n}{\ln n} \qquad (1)$$

tends to ∞ . That is why the statistical models in the thermodynamic limit, determined on the Cayley tree may be considered as "infinite-dimensional" [2]. If the site variables σ_i range over the discrete values 1,2 ... Q and the external magnetic field is directed along one value, say $\sigma = 1$, the Hamiltonian of the model is given by the expression

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j) - H\sum_i \delta(\sigma_i,1) \qquad (2)$$

where J is the interaction constant of the nearest neighbouring spins (below we assume that $J>0$), $\delta(\sigma_i,\sigma_j)=1$ if $\sigma_i=\sigma_j$, $\delta(\sigma_i,\sigma_j)=0$, otherwise. The first summation goes over all edges of the graph, the second one - over all sites.

The partition function of the model is

$$Z = \sum_{\{\sigma\}} P(\sigma) = \sum_{\{\sigma\}} \exp\left\{K\sum\delta(\sigma_i,\sigma_j) + h\sum_i \delta(\sigma_i,1)\right\} \qquad (3)$$

where

$$K = \frac{J}{kT}, \qquad h = \frac{H}{kT} \qquad (4)$$

and the summation in (3) goes over all spin configurations of the system. Let us determine the magnetization per site

$$M(\sigma_i) = \sum_{\{\sigma\}} \delta(\sigma_i,1)\, P(\sigma)/Z \qquad (5)$$

as well as the more convenient for the future variable - an order parameter

$$m = \frac{QM-1}{Q-1} \qquad (6)$$

which takes the value $m = 1$ in fully ordered state ($M = 1$), and $m = 0$ in fully disordered state ($M=1/Q$) of the system.

When "cutting apart" the Cayley tree in the central point, it separates into q identical branches (Fig.1). Then the partition function may be written in the form

$$Z = \sum_{\{\sigma_0\}} \exp\{h\delta(\sigma_0,1)\}\,[g_n(\sigma_0)]^q \qquad (7)$$

where σ_0 is the spin value in the central site, and equation

$$g_n(\sigma_0) = \sum_{\{s\}} \exp\left\{K\sum_{\langle ij\rangle}\delta(s_i,s_j) + K\delta(s_i,\sigma_0) + h\sum_i \delta(s_i,1)\right\} \qquad (8)$$

is, in fact, the partition function of an individual branch. Magnetization in the central point is expressed by

$$M_n(\sigma_0) = z^{-1}\sum_{\sigma_0}\delta(\sigma_0,1)\exp\{h\delta(\sigma_0,1)\}[g_n(\sigma_0)]^q \qquad (9)$$

The external field H singles out "the first direction". The symmetry between the other "directions" is kept, therefore two functions are enough to determine the state of the system:

$$g_n(\sigma_0=1)\equiv g_n(+)\,;\quad g_n(\sigma_0\neq 1)\equiv g_n(-);\quad x_n\equiv\frac{g_n(-)}{g_n(+)} \qquad (10)$$

The order parameter (6) in the new variables is equal to

$$m_n = \frac{e^h - x_n^q}{e^h+(Q-1)x_n^q} \qquad (11)$$

One can obtain recursion equations for x_n . It is not hard to see that

$$x_n = y(x_{n-1}) \qquad \text{where} \quad y = \frac{e^h+(e^K+Q-2)x^{q-1}}{e^{K+h}+(Q-1)x^{q-1}} \qquad (12)$$

Eq. (12) can be rewritten in a more convenient form

$$h = (q-1)\ln x + \ln[(Q-1)x-(z+Q-2)] - \ln(1-zx) \qquad (13)$$

where $z = e^K$ and $1/z < x < \frac{z+Q-2}{Q-1}$

It is seen from (13) that at high temperatures $h = h(x)$ is a monotonically decending function, but beginning from a certain value of $T = T_c^{(1)}$ the derivative $\frac{dh}{dx}$ becomes positive for definite x , i.e. the temperature $T_c^{(1)}$ is the phase transition point, as above it ($T > T_c^{(1)}$) the function $x = x(h)$ is single--valued (one stable point) and below it ($T < T_c^{(1)}$) it becomes multiple-valued (2 stable points). At $T = T_c^{(1)}$ the function $h(x)$ has a point of inflection, hence $T_c^{(1)}$ can be determined from

$$\frac{dh}{dx} = 0\,; \qquad \frac{d^2h}{dx^2} = 0 \qquad (14)$$

Solving this system one obtains

$$z_c^{(1)} = \frac{1}{2}\left[2-Q+\frac{1}{q-2}\sqrt{Q^2q^2-4(q-1)(Q-2)^2}\right. \qquad (15)$$

The phase transition point value x can also be found from (14) which is equal to

$$x_c = \frac{q}{q-2} \cdot \frac{1}{z_c} \tag{16}$$

As it is seen, when $Q > 2$, then $h_c \neq 0$, i.e. the phase transition takes place in non-zero field. The value x (hence the magnetization too) changes continuously, whence one can assume that at $T = T_c^{(1)}$ a transition of the II order takes place in the system. Below $T_c^{(1)}$ the equation (12) has two stable solutions and at the change of the field h the variable x jumps from one point to another. The value of h at which this jump takes place, decreases with the temperature and below $T = T_c^{(2)}$ the jumping takes place at zero field. This temperature

$$z_c^{(2)} = \frac{q+Q-2}{q-2} \quad ; \qquad T_c^{(2)} < T_c^{(1)} \tag{17}$$

is similar to the one obtained in Ref.[3] for the Cayley tree, and differs from the result of the mean field approximation[4]. When $Q > 2$ a phase transition of the I order takes place at $T = T_c^{(2)}$. The variable x changes step-wise and a spontaneous magnetization appears in the system. Note, that at $Q = 2$ (Ising model) both critical temperatures coincide:

$$K_c^{(1)} = K_c^{(2)} = \ln \frac{q}{q-2} \; ; \qquad h_c = 0 \tag{18}$$

According to scaling hypothesis, for every system having a II order phase transition, near the critical point the following approximation must be fulfilled:

$$h = M\,|M|^{\delta-1}\, h_s(t\,|M|^{-1/\beta}) \tag{19}$$

Here $t = \frac{T-T_c}{T_c}$ and $h_s(x)$ are certain homogeneous functions of an argument, named scaling function, δ and β are the critical exponents of the system. The scaling hypothesis (19) is written for the case of $h_c = m_c = 0$. Its generalization for our case is obvious:

$$\Delta h = \Delta m |\Delta m|^{\delta-1} h_s(t|\Delta m|^{-1/\beta}) \tag{20}$$

Let us denote

$$x_c = \ell^a \qquad x = \ell^{a+s} , \tag{21}$$

Near the critical point $s \ll 1$ the value of Δh can also be expanded in the degrees of s . With regard to the formula (13) this expression up to the third order will have the form

$$\Delta h = (q-1+A+B)s + [\tfrac{A}{2}(1-A) + \tfrac{B}{2}(1+B)]s^2 +$$
$$+[A(\tfrac{1}{6} - \tfrac{A}{2} + \tfrac{A^2}{3})] + B(\tfrac{1}{6} + \tfrac{B}{2} + \tfrac{B^2}{3})]s^3 + O(s^4) + \ldots \tag{22}$$

where $A = \dfrac{(Q-1)\ell^a}{(Q-1)\ell^a - \ell^K - Q + 2}$; $\qquad B = \dfrac{\ell^{K+a}}{1-\ell^{K+a}}$

The value of Δm can also be expanded in the degrees of s . As it is seen from (22) near the critical point $\Delta h \ll s$, hence $m(h,s) \approx m(h_c, s)$. Then substituting (21) into (11) one obtains

$$\Delta m \approx -\frac{Qq}{4(Q-1)} s \qquad \text{or} \qquad s \approx -\frac{4(Q-1)}{Qq}\Delta m \tag{23}$$

Substituting the value s from (23) into (22), let us write the final expression

$$H/kT_c = \Delta m^3 h_c(t/\Delta m^2) \tag{24}$$

where the scaling function $h_s(x)$ is equal to

$$h_3(x) = \frac{q-2}{q^2 Q}\left\{[q^2(Q-1)+(q-2)^2 Z_c^2] K_c x + \frac{16}{3}\cdot\frac{(q-1)(Q-1)^3}{Q^2}\right\} \tag{25}$$

As it is seen, the scaling hypothesis is confirmed, and the critical exponents are equal to $\beta = 1/2$; $\delta = 3$. They are independent of the system parameter Q and thus, coincide with the corresponding exponents in the Ising model on a Bethe lattice [2]. They also coincide with the values obtained by the mean-field approximation. The other critical exponents, in particular, $\alpha = 0$, can be calculated by application of the scaling hypothesis. This implies, that the given phase transition is of the II order.

The II order phase transitions we found, have a characteristic peculiarity which consists in the fact that the transition takes place at a non-zero field. The external field a priori breaks the $Z(Q)$ symmetry of the system. At the first sight it is not clear what particular symmetry is broken in consequence of phase transition, as the common feature of the II order transitions is the symmetry-breaking phenomenon [5].

However, the considered model is not only $Z(Q)$-invariant, but also is invariant under $Q!$ permutation group, which significantly differentiates between the Potts model and the planar $Z(Q)$ models. That is why the external field breaks only $Z(Q)$ symmetry, leaving the $(Q-1)!$ permutation group symmetry unbroken. Therefore, the II order transitions in the external field are, apparently, possible just in the models with $Q>2$. Such reasoning explains the separation of the Ising model ($Q = 2$) where the transition takes place at $h = 0$.

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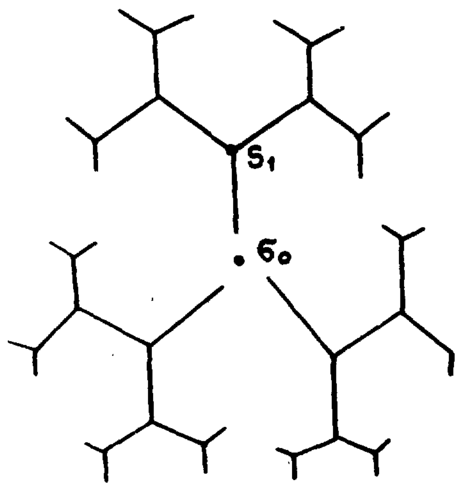

Fig.1. The Cayley tree with coordination number $q = 3$ and the number of shells $n = 4$. σ_0 is the value of the spin in the central site.

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КРИТИЧЕСКИЕ СВОЙСТВА МОДЕЛИ ПОТТСА НА РЕШЕТКЕ БЕТЕ

(на **английском языке**,перевод Папяна Г.А.)

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