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DIRAC AND WEYL FERMION DYNAMICS ON TWO-DIMENSIONAL SURFACE

ЦНИИатоминформ

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ԴԻՐԱԿՅԱՆ ԵՎ ՎԵՅԼԵՎՅԱՆ ՖԵՐՄԻՈՆՆԵՐԻ ԴԻՆԱՄԻԿԱԿԱՆ ԵՐԿՉԱՓ ՄԱԿԵՐԵՎՈՒՅԹԻ ՎՐԱ

Հետազոտվում են ֆերմիոնները երկչափ մակերևույթի վրա, որոնք փետեղված են եռաչափ տարածությունում: Վերապարամետրիզացումների նկատմամբ անփոփոխ կարգավորման դետերմինանտը որոշվում է համաչափության տարականոնությամբ /անոմալիայով/, ինչպես նաև Վես-Զումինոյի գործողությանը հանգեցնող Լորենց տարականոնությամբ, որի կառուցվածքը որոշվում է Հոպֆի տոպոլոգիական ինվարիանտով:

Երևանի ֆիզիկայի ինստիտուտ
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*) Միջուկային հետազոտությունների և միջուկային էներգետիկայի ինստիտուտ Բուլղարիայի ԳԱ, Սոֆիա, ԲԺՀ

In recent years a common analysis of topological properties of gauge field configurational space and their connection with the anomalies of classical theories [1-5] is made. Anomalies of currents, resulting from quantum corrections determine a term in effective lagrangian corresponding to breaking of the given symmetry. Wess-Zumino term of the effective lagrangian provides the breaking of chiral symmetry. This term has a topological meaning. In Ref.[6] Wess-Zumino actions are constructed for nonlinear 6-models on uniform spaces G/H in arbitrary dimensions.

In 2-dimensional models of chiral fermions in external fields the effective lagrangians are completely determined by gauge [7] or conformal and Lorentz anomalies [8,9].

In the present work fermions (Dirac and Weyl) on 2-dimensional surface, embedded in a 3-dimensional space, are investigated (the difference between Euclidean and Minkovski spaces is insignificant here). The Dirac action induced on 2-dimensional surface has reparametrization and conformal invariances, and also is invariant under the local rotations of tangent vectors of the surface (local Lorentz invariance). The local Lorentz invariance is violated by the use of reparametrizational-invariant regularization (according to the results of Refs [2,3,8,10]) which leads to the Wess-Zumino term, the structure of which at $d = 3$ is determined by the Hopf invariant of $S^3 \to S^2$ map. The conformal invariance is also broken leading to the Liouville action.

The Dirac action, induced on a 2-dimensional surface embeded in a 3-dimensional space, is determined by lagrangian

$$L = \frac{i}{2} g^{\alpha\beta} \bar{\Psi} (\gamma_\alpha \overrightarrow{\partial_\beta} - \overleftarrow{\partial_\beta} \gamma_\alpha) \Psi =$$

$$= i g^{\alpha\beta} \bar{\Psi} \gamma_\alpha (\partial_\beta + \Gamma_\beta) \Psi , \quad \alpha = 1, 2, \tag{1}$$

where $\gamma_\alpha = \partial_\alpha x^\mu \gamma_\mu$ are the induced Dirac γ-matrices, $g_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu$ is the induced metrics,

$$\Gamma_\alpha = \frac{1}{4} (\gamma^\beta \nabla_\alpha \gamma_\beta + n \partial_\beta n) \tag{2}$$

is the induced spinor connection, $n = n^\mu \gamma_\mu$, n^μ is the normal vector of the surface. The lagrangian (1) is invariant under local rotations of tangent vectors around the normal vector and under simultaneous rotation of spinors

$$\partial_\alpha x^\mu \to \Lambda^\mu{}_\nu \partial_\alpha x^\nu , \quad \Psi \to S \Psi , \quad \bar{\Psi} \to \bar{\Psi} S^{-1}, \tag{3}$$

where $\Lambda^\mu{}_\nu \in SO(2)$, S is an operator corresponding to rotation in spinor presentation.

Introducing tetrads e^α_a , $e_{a\alpha} e^a{}_\beta = g_{\alpha\beta}$, let us write the lagrangian (1) in tetrad formalizm:

$$\begin{aligned} L &= i \bar{\Psi} \gamma^a e^\alpha_a (\partial_\alpha + \Omega^{-1} \partial_\alpha \Omega - \frac{i}{2} n \omega_\alpha) \Psi = \\ &= i \bar{\Psi} \Omega^{-1} \sigma^a e^\alpha_a (\partial_\alpha - \frac{i}{2} \sigma_3 \omega_\alpha) \Omega \Psi = i \bar{\Psi} D \Psi , \\ &\Omega^{-1} \partial_\alpha \Omega = \frac{1}{4} (\gamma^a \partial_\alpha \gamma_a + n \partial_\alpha n) . \end{aligned} \tag{4}$$

Here $\gamma^a = e^a{}_\alpha \gamma^\alpha$, $\omega_\alpha = \frac{1}{2} \frac{\varepsilon^{\beta\lambda}}{\sqrt{g}} e^a{}_\beta \nabla_\lambda e_{a\alpha}$ is the standard spinor connection, σ^a, σ_3 are the Pauli matrices. The matrix Ω rotates from σ^a and σ_3 to γ^a and n

$$\gamma^a = \Omega^{-1}\sigma^a\Omega, \qquad n = \Omega^{-1}\sigma_3\Omega. \tag{5}$$

The action (4) is invariant under the conformal transformations

$$e^\alpha_a \longrightarrow \lambda^{-1} e^\alpha_a, \qquad \psi \longrightarrow \lambda^{1/2}\psi. \tag{6}$$

It is easy to check that operator D (4) anticommutates with the matrix n, which allows to determine chiral fermions:

$$\psi_{L,R} = \frac{1 \pm n}{2}\psi. \tag{7}$$

Reparametrization invariance allows one to choose $e^a{}_\alpha$ in the form

$$e^\alpha_a = \frac{1}{\sqrt{\rho}}\delta^\alpha{}_a .$$

Then Dirac operators of chiral fermions have the form

$$\begin{aligned} D_L &= \gamma^+\Omega^{-1}(\rho^{-1/2}\partial_+ - \tfrac{1}{2}\partial_+\rho^{-1/2})\Omega = \gamma^+\mathcal{D}_+ \\ D_R &= \gamma^-\Omega^{-1}(\rho^{-1/2}\partial - \tfrac{1}{2}\partial_-\rho^{-1/2})\Omega = \gamma\mathcal{D}_- , \end{aligned} \tag{8}$$

The effective action of chiral fermions formally is equal to

$$W = \ln\det D_L . \tag{9}$$

Let us calculate the variation of W under the change of independent variables ρ and Ω

$$\delta W = \mathrm{Tr}\,\delta D_L D_L^{-1} = \mathrm{Tr}\,\delta D_L D_R\, D_R^{-1} D_L^{-1} \tag{10}$$

Regularizing this expression by the proper time cut-off one obtains

$$\delta W = \mathrm{Tr} \int_{\varepsilon}^{\infty} dt \;\; \delta D_L D_R \, e^{-t D_L D_R} , \tag{11}$$

The operators $\mathcal{D}\pm$, determined in (8), satisfy the following relations:

$$\mathcal{D}_\alpha \gamma_\beta = \gamma_\beta \widetilde{\mathcal{D}}_\alpha \;\; , \;\; \mathcal{D}_\alpha n = n \mathcal{D}_\alpha , \tag{12}$$

where $\widetilde{\mathcal{D}}_\alpha$ differs from $\mathcal{D}_\alpha$ by the sign before $\partial \pm \rho^{-1/2}$. Therefore

$$e^{-tD_L D_R} = e^{-2t \frac{\gamma^+\gamma^-}{2} \widetilde{\mathcal{D}}_+ \mathcal{D}_-} =$$

$$= \frac{\gamma^+\gamma^-}{2} e^{-2t \widetilde{\mathcal{D}}_+ \mathcal{D}_-} + \left(1 - \frac{\gamma^+\gamma^-}{2}\right) \tag{13}$$

The second term does not contribute to (12) because of the property $D_R \gamma^- = 0$.

Using (8) one will obtain for δD_L

$$\delta D_L = \delta \ln \rho^{-1/2} D_L + D_L \Omega^{-1} \delta\Omega - \Omega^{-1} \delta\Omega D_L , \tag{14}$$

Substituting (14) into (11) and using (12), (13) one obtains the equation

$$\delta W = \lim_{\varepsilon \to 0} \mathrm{Tr} \Big\{ \Omega^{-1} \delta\Omega \frac{\gamma^-\gamma^+}{2} e^{-2\varepsilon \widetilde{\mathcal{D}}_- \mathcal{D}_+} - \Omega^{-1} \delta\Omega \frac{\gamma^+\gamma^-}{2} e^{-2\varepsilon \widetilde{\mathcal{D}}_+ \mathcal{D}_-} + \delta \ln \rho^{-1/2} e^{-2\varepsilon \widetilde{\mathcal{D}}_- \mathcal{D}_+} \Big\} , \tag{15}$$

Here $\lim\limits_{\varepsilon \to 0} e^{-2\varepsilon \widetilde{\mathcal{D}}_- \mathcal{D}_+}$ and $\lim\limits_{\varepsilon \to 0} e^{-2\varepsilon \widetilde{\mathcal{D}}_+ \mathcal{D}_-}$ are the zero Seeley coefficients of operators $\widetilde{\mathcal{D}}_- \mathcal{D}_+$ and $\widetilde{\mathcal{D}}_+ \mathcal{D}_-$ which can be calculated using the formulae presented, for instance, in Ref. [11]

$$\lim_{\varepsilon\to 0} e^{-2\varepsilon\widetilde{\mathcal{D}}_- \mathcal{D}_+} = \lim_{\varepsilon\to 0} e^{-2\varepsilon\widetilde{\mathcal{D}}_+ \mathcal{D}_-} =$$
$$= \frac{1}{48\pi}\sqrt{g}\, R + \sqrt{g}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}, \tag{16}$$

where R is the Gaussian curvature of the 2-dimensional surface. The divergent term in (16) determines the local polynomial over ρ and Ω, which can be neglected in regularized W.

Using relation $[\gamma^+, \gamma^-] = 2n$, one obtains for the regularized W

$$\delta W = \frac{1}{48\pi}\left[\mathrm{Tr}\,\Omega^{-1}\delta\Omega n\sqrt{g}R + \delta \ln\rho\sqrt{g}\,(R+\mu^2)\right]. \tag{17}$$

Equation (17) shows that the conformal and Lorentz symmetries of classic lagrangian are broken by quantum fluctuations. So, one has

$$\begin{aligned} \langle T^{\alpha}{}_{\alpha}\rangle &= -\frac{1}{48\pi}\sqrt{g}\,(R+\mu^2) \\ \langle \nabla_\alpha J^\alpha_L\rangle &= -\frac{i}{48\pi}\sqrt{g}\,R\,. \end{aligned} \tag{18}$$

Integration of the second term in (17) gives the Liouville action [12]

$$W_{Liunville} = \frac{1}{192\pi}\int d^2\xi\left[(\partial_\alpha \ln\rho)^2 + \mu^2\rho\right]. \tag{19}$$

It will be shown now that the integration of the first term gives the Wess-Zumino action which is determined by the Hopf invariant of the map $\Omega : S^3 \longrightarrow S^2$.

Let us consider the 2-form ω_2 on S^2

$$\omega_2 = \frac{1}{8\pi}\,\mathrm{tr}\; n\, dn\wedge dn \tag{20}$$

The map $\Omega : S^3 \to S^2$ induces the exact 2-form on S^3 :

$$f^*(\omega_2) = \frac{1}{8\pi} tr\, n\, \partial_\mu n\, \partial_\nu n\, dx^\mu \wedge dx^\nu , \tag{21}$$

where $\mu, \nu = 1,2,3$, x^μ are the coordinates on S^2 . To show the exactness of the form $f^*(\omega_1)$, let us make the following transformation:

$$n = \Omega^{-1} \sigma_3 \Omega , \tag{22}$$

The matrix Ω is determined up to the left multiplication by the matrix of rotation around n . Substituting (22) into (21) one has

$$\begin{aligned} f^*(\omega_2) &= \frac{1}{2\pi} tr\, \sigma_3\, d\Omega \wedge d\Omega^{-1} = \\ &= \partial_\mu A_\nu \; dx^\mu \wedge dx^\nu = d\omega_1 , \end{aligned} \tag{23}$$

where

$$A_\mu = \frac{1}{2\pi} tr\, \sigma_3 \;\Omega\, \partial_\mu \Omega^{-1} . \tag{24}$$

The invariance $f^*(\omega_2)$ under the local rotations around the normal vector manifests itself in the fact that A_μ transforms like the $U(1)$-gauge field.

Really, for the infinitesimal transformation $\Omega \to \Omega + \delta\Omega = \Omega S$

$$\begin{aligned} A_\mu &= -\frac{1}{2\pi} tr\, n\, \Omega^{-1} \partial_\mu \Omega \to A_\mu + \delta A_\mu \\ \delta A_\mu &= -\frac{1}{2\pi} tr\, n\, \partial_\mu S \cdot S^{-1} . \end{aligned} \tag{25}$$

Since only the n -containing part of $\Omega^{-1} \partial_\mu \Omega$ contributes to A_μ (24), it is not difficult to see that

$$\delta A_\mu = -\frac{1}{2\pi}\partial_\mu \mathrm{tr}\, n\, \Omega^{-1}\delta\Omega \, , \tag{26}$$

It follows from above that the value

$$H = \int_{S^3} f^*(\omega_2)\wedge\omega_1 = \int A_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} d^3x \tag{27}$$

is invariant under any variations of Ω . H is called Hopf invariant.

The Wess-Zumino action, being an integral of the first term in (17), is determined in the usual way [13] . Integrating the Hopf integral density over the hemisphere S^3 of boundary S^2 one obtains

$$W_{Hopf} = \frac{i\pi}{6}\int_{\partial D = S^2} A_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} d^3x . \tag{28}$$

Using the relations (24) and (26) it is easy to show that the variation δW_{Hopf} is equal to the first term in (17). For an effective action of W one obtains

$$W = -\frac{1}{192\pi}\int d^2\xi\, [(\partial_a \ln\rho)^2 + \mu^2\rho] + \frac{i\pi}{6}\int d^3x\, A_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} .$$

In conclusion, we have calculated the effective action for the induced Dirac theory of chiral fermions on 2-dimensional surface embedded in 3-dimensional space. It is given by the expression

$$W_{L,R} = \frac{1}{192\pi}\int d^2\xi \sqrt{g}\, R\, \Delta^{-1} R + \mu^2 \int d^2\xi \sqrt{g} \pm \frac{i\pi}{6}\int d^3x\, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \, , \tag{29}$$

where R is the intrinsic curvature, Δ is the scalar Laplacian, the field A_μ is determined in Eq.(24). There is no Lorentz anomaly for Dirac fermions and thus, the effective action will contain only the Liouville term with a coefficient two times larger than that in the action for chiral fermions.

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ДИНАМИКА ДИРАКОВСКИХ И ВЕЙЛЕВСКИХ ФЕРМИОНОВ НА ДВУХМЕРНОЙ ПОВЕРХНОСТИ

(на английском языке, перевод Г.А.Папяна)

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