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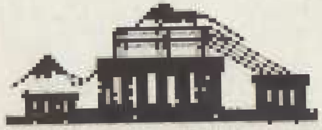

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O.M.KHUDAVERDYAN, R.L.MKRTCHYAN

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INTEGRAL INVARIANTS OF BUTTIN'S BRACKET

We study densities (more general objects) which may be integrated over surfaces in the superspace, invariant with respect to the superdiffeomorphism transformations which do not change the Buttin's bracket. The only such nontrivial object is, in a definite sense, the odd nondensity explicitly constructed here.

Yerevan Physics Institute
Yerevan 1986

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ИНТЕГРАЛЬНЫЕ ИНВАРИАНТЫ СКОБКИ БУТИНА

В работе исследуется возможность существования плотностей – наиболее общих объектов интегрирования по суперповерхностям, инвариантных относительно суперканонических преобразований, сохраняющих скобку Бутина. Строится инвариантная плотность второго ранга.

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Ереван 1988

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Some interest in the properties of Buttin's bracket has been arisen recently in theoretical physics [1,2]. In this work we study one of its properties, namely the problem of constructing its integral invariants, or, more generally, the densities of arbitrary weight.

Let $E^{n,n}$ be the superspace with coordinates $(x^1,\dots,x^n,\theta^1,\dots,\theta^n)$ and f, g be functions on $E^{n,n}$, having definite parities $p(f)$, $p(g)$. The Buttin bracket $\{f,g\}_B$ of these two functions is defined as [3]

$$\{f,g\}_B = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta^i} - (-1)^{p(f)} \frac{\partial f}{\partial \theta^i} \frac{\partial g}{\partial x^i} \right) \qquad (1)$$

Derivatives are the left ones.

We define supercanonical transformations of second kind as all transformations of $E^{n,n}$ which don't change the brackets (1), or, equivalently, which don't change the odd nondegenerate closed two-form [4,5]

$$\omega_2 = \sum_{i=1}^{n} dx^i d\theta^i$$

Supercanonical transformations of the first kind are transformations of $E^{2n,N}$, which don't change even nondegenerate closed two-form

$$\omega_{PM} = \sum_{1\le i\le m} dx^i \wedge dx^{i+m} + \sum_{1\le i\le n} \varepsilon_i (d\theta^i)^2, \quad \varepsilon_i = \pm 1 \qquad (2)$$

or, equivalently, which don't change the Poisson-Martin bracket $\{\ ,\ \}_{PM}$ — the straightforward generalization of Poisson bracket:

$$\{f,g\}_{PM} = \sum_{1\le i\le m}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^{i+m}} - \frac{\partial f}{\partial x^{i+m}}\frac{\partial g}{\partial x^i}\right) +$$

$$+ \sum_{1\le i\le n} (-1)^{p(f)} \varepsilon_i \frac{\partial f}{\partial \theta^i}\frac{\partial g}{\partial \theta^i}$$

Supercanonical transformation of the first and second kind are the only ones, maintaining nondegenerate closed two-form [5]. The latter in appropriate coordinates coincides with ω_B or ω_{PM}.

The integral invariants of supercanonical transformations of the first kind are constructed in Ref. [7]. They are a natural generalization of classical Poincaré-Cartan integral invariants and coincide with them, when the number of odd coordinates θ^i is zero. Namely, let $\Omega^{2m,n}$ be a $(2m,n)$-dimensional surface in $E^{2m,n}$, given in a definite parametrization by the $Z^A = Z^A(\xi^K)$

$$Z^A = (x^1,\dots,x^{2m},\theta^1,\dots,\theta^n)$$
$$\xi^K = (\xi^1,\dots,\xi^{2m},\nu^1,\dots,\nu^n)$$

Then the quantity

$$\Phi(\Omega) = \int \sqrt{\operatorname{Ber}\frac{\partial Z^A}{\partial \xi^K}\,\Gamma_{AA'}\,\frac{\partial Z^{A'}}{\partial \xi^{K'}}}\; d\nu_\xi \qquad (3)$$

$$\omega_{PM} = dZ^A\,\Gamma_{AA'}\,dZ^{A'}$$

a) is invariant with respect to supercanonical transformations of first kind, b) is reparametrization invariant, c) at n = 0 it coincides with the well-known Poincaré-Cartan integral invariant:

$$\Phi(\Omega) = \int_{(M=2m)} \underbrace{\omega_{PM}\wedge\dots\wedge\omega_{PM}}_{(m\ \text{times})}$$

Note, that (3) is defined for the surfaces $\Omega^{2m,n}$ on which the restriction of fundamental two-form (2) is nondegenerate. We impose a similar restriction on the surfaces also in the case of supercanonical transformations of the second kind. This means, in particular, that in the latter case we shall consider only (m,m)-dimensional surfaces $\Omega^{m,m}$.

Evidently, the direct generalization of (3) for the case of Buttin's bracket (1) is senseless [8].

Now we give the exact definition and formulation of our results.

Consider the integral

$$\Phi_H(\Omega) = \int H\left(Z^A(\xi^K), \frac{\partial Z^A}{\partial \xi^K}, \dots, \frac{\partial^r Z^A}{\partial \xi^{K_1}\dots\partial \xi^{K_r}}\right) d\nu_\xi \qquad (4)$$

where $Z^A = (x^1,\dots,x^m,\theta^1,\dots,\theta^m)$ are coordinates in the superspace $E^{N.M}$, $\zeta^K = (\xi^1, \dots, \xi^n, \theta^1, \dots, \theta^m)$ are coordinates of subspace $E^{n.m}$, $z^A(\zeta^K)$ is parametrization of (n.m)-dimensional surface $\Omega^{n.m}$ in $E^{N.M}$. $A(z^A, \frac{\partial z^A}{\partial \zeta^K}, \dots, \frac{\partial^k z^A}{\partial \zeta^{K_1} \dots \partial \zeta^{K_k}})$ is called (k, N.M)(n.m)-density of rank k and weight ρ ($\rho \neq 0$) if under reparametrization $\zeta^K = \zeta^K(\tilde\zeta)$, $\tilde z^A(\tilde\zeta) = z^A(\zeta(\tilde\zeta))$ it changes in the following way:

$$A\left(\tilde z^A(\tilde\zeta), \frac{\partial \tilde z^A(\tilde\zeta)}{\partial \tilde\zeta^K}, \dots, \frac{\partial^k \tilde z^A(\tilde\zeta)}{\partial \tilde\zeta^{K_1} \dots \partial \tilde\zeta^{K_k}}\right) = A\left(z^A(\zeta), \frac{\partial z^A(\zeta)}{\partial \zeta^K}, \dots, \frac{\partial^k z^A(\zeta)}{\partial \zeta^{K_1} \dots \partial \zeta^{K_k}}\right)\left(\mathrm{Ber}\, \frac{\partial \zeta}{\partial \tilde\zeta}\right)^{\rho} \quad (3)$$

For $\rho = 1$ integral (4) doesn't depend on the particular choice of parametrization of $\Omega^{n.m}$. If density A of weight $\rho \neq 1$ is even and its number part is nonzero, then $(A)^{1/\rho}$ is well-defined and represents itself the density of weight $\rho = 1$. In other cases, for example for odd densities, the weight cannot be changed and it is an essential characteristic of the density.

We can describe the new densities in the language of the so-called D-densities [illegible]. The latter are relevant when surfaces $\Omega^{n.m}$ are given by the equations $F^\alpha = 0$

$$F^\alpha = (f^1, \dots, f^{N-n}, \varphi^1, \dots, \varphi^{M-m})$$

f^i are even, φ^α are odd functions on $E^{N.M}$. In this case (4) is substituted by

$$\Phi_{\check A}(\Omega) = \int \check A\left(z^A, \frac{\partial F^\alpha}{\partial z^A}, \dots, \frac{\partial^k F^\alpha}{\partial z^{A_1} \dots \partial z^{A_k}}\right) \delta(F^\alpha)\, dv_z \quad (5)$$

and the definition of the D-density of the weight ρ is, that when using two equivalent equations

$$F'^\alpha = 0, \quad F'^\alpha = \eta^\alpha_\beta(z^A) F^\beta$$

instead of $F^\alpha = 0$, the D-density $\check A$ must satisfy

$$\check A\left(z^A, \frac{\partial F'^\alpha}{\partial z^A}, \dots, \frac{\partial^k F'^\alpha}{\partial z^{A_1} \dots \partial z^{A_k}}\right) = \check A\left(z^A, \frac{\partial F^\alpha}{\partial z^A}, \dots, \frac{\partial^k F^\alpha}{\partial z^{A_1} \dots \partial z^{A_k}}\right)\left(\mathrm{Ber}\, \eta^\alpha_\beta\right)^{\rho} \quad (6)$$

The D-density $\check A$ corresponding to density A is given in the particular case when

$$F^\alpha(z^A) = y^\alpha - f^\alpha(\zeta^K), \quad z^A = (y^\alpha, \zeta^K)$$

$$z^A(\zeta^K) = \begin{cases} y^\alpha(\zeta^K) = f^\alpha(\zeta^K) \\ \zeta^K(\zeta^K) = \zeta^K \end{cases} \quad (7)$$

by the

$$\check A\left(z^A, \frac{\partial F^\alpha}{\partial z^A}, \dots, \frac{\partial^k F^\alpha}{\partial z^{A_1} \dots \partial z^{A_k}}\right) = A\left(z^A, \frac{\partial z^A}{\partial \zeta^K}, \dots, \frac{\partial^k z^A}{\partial \zeta^{K_1} \dots \partial \zeta^{K_k}}\right) \quad (8)$$

It is evident that, for $\rho = 1$, if D-density $\check A$ corresponds to the density A, then (4) and (5) coincide

$$\Phi_{\check A}(\Omega) = \Phi_A(\Omega) \quad (9)$$

For example, integral invariant (3) corresponds to

$$\Phi_H(D) = \int \sqrt{\operatorname{Ber}\,\{F^{\sigma}, F^{\sigma'}\}_{\ldots}}\ \prod_{\sigma} \delta(F^{\sigma})\, dv_x \qquad (10)$$

and eq.(9) holds.

The important statement is, that correspondence between densities and I-densities, established by [7],[8] in particular coordinates, is really independent of coordinates and depends only on the volume form on $E^{M,N}$. Namely, the correspondence $\rho \to \hat{\rho}$ holds also if we use the coordinates $\tilde{z}^A$ instead of $z^A = (x^i, \theta^i)$, provided

$$\operatorname{Ber} \frac{\partial \tilde{z}^A}{\partial z^{A'}} = 1 \qquad (11)$$

i.e. if the volume form remains unchanged.

At $g = 1$ this statement follows immediately from (9). In general case, this statement may be checked by straightforward calculations.

Our main result is the

Theorem. Let we have the superspace $E^{M,N}$, G is the group of supercanonical transformations of second kind, i.e. the transformations which don't change the Buttin bracket [1], G_0 is the subgroup of G consisting of transformations preserving the volume form, i.e. having the unit superJacobian (11). Then

a) there are no G-invariant (m, m)-densities for rank one there is only one (trivial) G_0-invariant density, it

is even (M, M)-density – the volume form.

b) For rank two there is only one G_0-invariant (m, m)-density ρ: it is (M−1, M−1) one, semidensity (i.e. with weight $g = 1/2$).

c) the corresponding I-density $\hat{\rho}$ is

$$\hat{\rho} = \frac{1}{\sqrt{\{f,\varphi\}_B}} \left(\sum_i \frac{\partial^2 f_0}{\partial x^i \partial \theta^i} - \{f_0, \{f, \varphi_0\}_B\}_B \right) \qquad (12)$$

$$f_0 = f + \frac{1}{2} \frac{\{f,f\}_B}{\{f,\varphi\}_B}\varphi ; \qquad \varphi_0 = \frac{1}{\{f,\varphi\}_B}\varphi$$

where the surface is defined by the equations $f = 0$, $\varphi = 0$, f and φ are even and odd functions on $E^{M,M}$, respectively.

Here the operator $\Delta = \sum_i \partial^2/\partial x^i \partial \theta^i$ is G_0-invariant (but not G-invariant, see Ref.[2]) so (12) is G_0-invariant and not G-invariant. The property [5] for (12) is checked directly. What follows is the brief proof of a), b).

The transformations from G are generated by the odd Hamiltonians, namely, to an odd function $H(z)$ on $E^{M,M}$ corresponds the one-parameter family of transformations of $E^{M,M}$ $z \to \tilde{z}(t,z)$, $t \in R$, through the equations

$$\tilde{z}(0,z) = z, \qquad \frac{d\tilde{z}(t,z)}{dt} = \{H, \tilde{z}\}_B \qquad (13)$$

H generates transformations from G_0 iff

$$\Delta H = 0 \qquad (14)$$

Using (13) and reparametrizations, it is easy to find a "standard" form for the expansion of functions $\Xi^k(\xi)$ determining the [illegible] (m, M) surface, near the ordinary point ξ_0 (which we may choose equal to zero). This form is:

$$x^\alpha = \xi^\alpha, \quad \theta^\alpha = \gamma^\alpha, \quad \alpha = 1, \dots, m$$

$$x^i = O(|\xi|^2) \qquad i = m+1, \dots, M \qquad (15)$$

$$\theta^i = O(|\xi|^2)$$

for $m \le M-2$, and

$$x^\alpha = \xi^\alpha, \quad \theta^\alpha = \gamma^\alpha$$

$$x^M = \psi \sum_1 \xi^\alpha \gamma^\alpha + O(|\xi|^2) \qquad (16)$$

$$\theta^M = O(|\xi|^2)$$

for $m = M-1$.

The proof is simple and we omit it. Note, that non-trivial term $\psi \sum_1 \xi^\alpha \gamma^\alpha$ in (16) may be reduced to zero by the transformation from S, but not from S_0.

Consider now the following transformation of $E^{M,M}$ belonging to S_0: $x^1 \to 2x^1$, $\theta^1 \to \frac{1}{2}\theta^1$, $x^M \to \frac{1}{2}x^M$, $\theta^M \to 2\theta^M$ and supply it by the subsequent reparametrization $\xi^1 \to 2\xi^1$, $\gamma^1 \to \frac{1}{2}\gamma^1$. Then (15) remains unchanged, and [illegible] in (16)

becomes 2ψ. Since the superposition of this reparametrization is E', it follows, that among invariant densities of rank two exist only one, with: $M-1$, $M-1$)-density, of weight 1/2, which for surface in standard parametrization (15) is equal to ψ. This density easily exists: corresponding density is given by (15), property (2) is satisfied with $\rho = 1/2$.

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