индекс 3624

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

Preprint ЕФИ-920(71)-86

ԵՐԵՎԱՆԻ ՖԻԶԻԿԱՅԻ ԻՆՍՏԻՏՈՒ
ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУ

V.G.GURZADYAN, A.A.KOCHARYAN

HYPERBOLICITY IN PSEUDO-RIEMANN SPACES

ЦНИИатоминформ

ЕРЕВАН-1986

Նախնատիպ ЕФИ-920(71)-86

Վ.Գ.ԳՈՒՐԶԱԴՅԱՆ, Ա.Ա.ՔՈՉԱՐՅԱՆ

ՀԻՊԵՐԲՈԼԱԿԱՆՈՒԹՅՈՒՆԸ ԹՎԱՑՅԱԼ-ՌԻՄԱՆՅԱՆ ՏԱՐԱԾՈՒԹՅՈՒՆՆԵՐՈՒՄ

Հետազոտվում են Թվացյալ-Ռիմանյան/Լորենցյան/ (−,+,...,+) նշանագրությամբ(signature) տարածաժամանակային բազմաձևությունների երկրաչափական հատկությունները: Տրված է հիպերբոլականության սահմանումը այդ տարածություններում, և գտնված են համապատասխան անհրաժեշտ պայմաններ՝ կիրառելի տիեզերագիտական խնդիրներում:

Երևանի Ֆիզիկայի ինստիտուտ
Երևան 1986

One of most fruitful directions of research in cosmology is the study of geometrical properties of the space-time manifold. Among classical achievements on this way are Hawking's and Penrose's theorems on the existence of singularities for large class of solutions of Einstein equations.

The initial point of the investigation in real space, as well as, in Wheller-DeWitt superspace, being the space of all three-metrics with signature (-+,+,+,+,+) [1] (see also [2]) is the study of dynamical systems in pseudo-riemann manifolds.

Below we shall study the global properties of these manifolds analogous to the results on the behaviour of close geodesics in the theory of dynamical systems [3-6]. The impossibility of direct application of results of that theory developed for Riemann spaces is evidently connected with the pseudo-riemann signature of real physical and Wheeler-DeWitt manifolds, e.g. it is easy to see that two close geodesics can diverge in pseudo-riemann space, while the value of divergence vector may remain close to zero. Therefore not only new criteria are needed here but one should also redifine the hyperbolicity property itself.

Let M be n-dimensional pseudo-riemann manifold with a metric g of Lorentzian signature (-,+,...,+) and Levi-Civita connection ∇. Consider a geodesic flow on M, i.e. a group of mapping $\{S^t\}$ of a space $T^\lambda M$, where

$$T^\lambda M = \{(x,u);\, x\in M,\, u\in T_x M,\, g(u,u) = \|u\|^2 = \lambda\},\quad \lambda = 0, \pm 1.$$

Each mapping S^t performs here a shift of linear element $\xi = (x,u)$ along defined geodesic on distance t.

At first it is necessary to generalize the definition of hyperbolicity for pseudo-riemann spaces.

Let $\gamma(t)$ be a geodesic on M passing by a point $x\in M$ and $\{E_a\}$ -fixed n-dimensional basis on $T_x M$. Transfering $\{E_a\}$ keeping parallel along $\gamma(t)$ and getting a basis in $T_x M$ at every t one thus defines a Fermi basis. Each vector $X\in T_{\gamma(t)}M$ can be presented by means of Fermi basis [7,8]

$$X(t) = X^a(t)E_a.$$

The expression

$$\|X\|^2_E = \sum_a (X^a)^2$$

defines E -the value vector X with respect to basis $\{E_a\}$. Let $\{E_{a'}\}$ be another basis in x. Then a non-singular matrix $\Phi_a^{\ell'}$ exist, such that

$$E_a = \sum_{\ell'} \Phi_a^{\ell'} E_{\ell'}.$$

So far as both $\{E_a\}$ and $\{E_{a'}\}$ being Fermi basises are transfered parallel along $\gamma(t)$, this relation must be satisfied for constant $\Phi_a^{\ell'}$. Thus we have

$$X^{\ell'}(t) = \sum_a \Phi_a^{\ell'} X^a(t).$$

From here taking into account the non-singularity of $\Phi_a^{\ell'}$ we come to the following inequality

$$c\sum_a (X^a)^2 \le \sum_{\ell'} (X^{\ell'})^2 \le c^{-1}\sum_a (X^a)^2$$

or

$$c\|X\|^2_E \le \|X\|^2_{E'} \le c^{-1}\|X\|^2_E, \qquad \text{(I)}$$

where c is a positive constant.

Now we are able to give the definition of hyperbolicity of geodesics(for notations see [3,6]).

Definition 1. Geodesic $\gamma_x(t) = S^t(\xi)$, $\|\dot\gamma_x(t)\|^2 = \lambda$ is a λ -hyperbolic one, if there exist subspaces

$$W^s(S^t(\xi)),\quad W^u(S^t(\xi))$$

and $W^0(S^t(\xi))$ of tangent space $T_{S^t(\xi)}T^\lambda M$ and numbers $A\ne 0$, $0<\mu<1$, such that

$$T_{S^t(\xi)}T^\lambda M = W^s(S^t(\xi)) \oplus W^u(S^t(\xi)) \oplus W^0(S^t(\xi)),$$

$$dS^\tau W^s(S^t(\xi)) = W^s(S^{t+\tau}(\xi)),\; dS^\tau W^u(S^t(\xi)) = W^u(S^{t+\tau}(\xi)),$$

where $W^0(S^t(\xi))$ is a one dimensional space determined by flow vector and evidently being invariant with respect to dS^t.

For each $t,\tau \geqslant 0$ and for a certain basis $\{E_a\}$ we have

$$\|dS^\tau v\|^2_E \le A^2\mu^{2\tau}\|v\|^2_E,\quad v\in W^s(S^t(\xi));$$

$$\| dS^{\tau} v \|^2_E \geqslant A^{-2} \mu^{-2\tau} \| v \|^2_E , \quad v \in W^u(S^t(\xi)),$$

where

$$\| v \|^2_E = \| d\pi_\lambda v \|^2_E + \| K \cdot v \|^2_E ,$$

$$v \in TT^\lambda M ; \quad \pi_\lambda : TT^\lambda M \to T^\lambda M ,$$

K is the mapping of connection ∇ [8].

One can see that this definition of hyperbolicity does not depend on the choice of basis $\{E_a\}$.

Definition 2. Geodesic flow is λ-hyperbolic of its each geodesic is λ - hyperbolic.

Our next aim is to obtain corresponding criteria of hyperbolicity property. For it let us first define Jacobi field $Y(t)$ along a geodesic $\gamma(t)$ determined by Jacobi equation [4,8]

$$\nabla_u \nabla_u Y + R(u, Y) u = 0 , \tag{2}$$

where R is the Riemann tensor.

Confront now each vector $v \in TT^\lambda M$ with a solution of Jacobi equation $Y(t)$ having initial conditions

$$Y_v(0) = d\pi v ,$$

$$\nabla_u Y_v(0) = K \cdot v .$$

The resulting mapping

$$f : v \to Y_v(t)$$

is an isomorphism and

$$d\pi \, dS^t v = Y_v(t) ,$$

$$K \, dS^t v = \nabla_u Y_v(t) . \tag{3}$$

One can see that Eq.(2) is an equation in variations for flow $\{S^t\}$. Using (3) we obtain

$$\| dS^t v \|^2_E = \| Y_v(t) \|^2_E + \| \nabla_u Y_v(t) \|^2_E . \tag{4}$$

Jacobi equation and (4) enables us to check the hyperbolity conditions (Definition 1).

Lyapunov characteristic number χ for maximal geodesics γ and vector v is defined as follows (cf. [4,6])

$$\chi(\gamma, v) = \limsup_{t \to \infty} \frac{\ln \| dS^t_\gamma v \|^2_E}{2t} .$$

Evidently $\chi(\gamma, v)$ does not depend on choice of $\{E_a\}$ too.

Definition 3. Geodesic γ is considered to be stable if $\forall \varepsilon > 0 , \exists \delta(\varepsilon) > 0$ such that from $\| v \|^2_E < \delta$ the condition $\| dS^t_\gamma v \|^2_E < \varepsilon$ follows for each t ; otherwise the geodesic γ is unstable.

These definitions are again invariant respect the choice of $\{E_a\}$. In view of this fact we can define convenient basis (cf. [7]).

For arbitrary geodesic $\gamma(t)$ from $T^{-1}M$ we choose an orthonormalized basis on point $\gamma(0)$ in a form

$$E_0 = \gamma(0) = u , \quad E_1 , \dots , E_{n-1} ;$$

$$g(E_a, E_b) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad (5)$$

where E^a is a dual basis.

If the following relations are satisfied

$$\nabla_u E_a = 0 = \nabla_u E^b ,$$

then the basis on $T^0 M$ can be

$$E_0 = u, \quad g(E_a, E_b) = \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & & & 0 \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{pmatrix} \qquad (6)$$

and on $T^1 M$:

$$E_0 = u, \quad g(E_a, E_b) = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} . . \qquad (7)$$

For the vector field

$$Y(t) = z^a(t) E_a$$

Jacobi equation

$$\frac{d^2 z^a}{dt^2} + R^a_{\ bcd}\, u^b u^d z^c = 0$$

can be rewritten in a form

$$\ddot{z}^a(t) + K^a_{\ b}(t)\, z^b(t) = 0, \qquad (8)$$

where

$$K^a_{\ b} = \langle E^a, R(u, E_b)u \rangle = R^a_{\ bcd}\, u^c u^d .$$

From (6) and Jacobi equation (8) it follows that

$$\dot{z}^1 = 0,$$

i.e. none of geodesic flows can be 0-hyperbolic.

Up to now we have considered geodesic flows on a manifold with Lorentzian signature (-+,...,+). It can be shown that the same results are valid in more general case of dynamical systems on a manifold with signature (-,...-+,...+).

The derived results enable one to study global geometrical properties of manifolds within concrete cosmological models. Thus based on these results we have shown the existence of exponential instability in Whiller-De-Witt superspace in the case of homogenous cosmological models[9].

References

1, De Witt B.S. Quantum Theory of Gravity. I. The Canonical Theory, Phys.Rev. 1967, vol.160, p.1113.

2. Hawking S.W. The Quantum State of the Universe, Nucl.Phys. 1984, vol.B239, p.257.

3. Anosov D.V. Geodesic Flows on a Compact Riemann Manifold of Negative Curvature, Proc. MIAN, 1967, vol.90.

4. Pesin Ya.B. Lyapunov Characteristic Numbers and Smooth Ergotic Theory, UMN, 1977, vol.XXXII, No.4, p.55-112.

5. Pesin Ya.B. Geodesic Flows with Hyperbolic Behavior of Trajectories and Related Objects, UMN, 1981, vol.XXXVI, No.4, p.3-51.

6. Modern Problems of Mathematics, vol.2, VINITI, 1985,Moscow (in Russian).

7. Hawking S.W., Ellis G.F.R. The Large Scale Structure of Space-Time, Cambridge University Press, 1973.

8. Gromoll D., Klingenberg W., Meyer W. Riemannsche Geometric in Grossen, Lectures Notes in Math. 55, Springer-Verlag, Berlin, 1975.

9. Gurzadyan V.G., Kocharyan A.A. Instability in Wheeler-De Witt Superspace, EPI-921(72), 1986, Yerevan.

The manuscript was received 17 June 1986

В.Г.ГУРЗАДЯН, А.А.КОЧАРЯН

ГИПЕРБОЛИЧНОСТЬ В ПСЕВДОРИМАНОВЫХ ПРОСТРАНСТВАХ

(на английском языке, перевод В.Г.Гурзадяна)

Редактор Л.П.Мукаян

Технический редактор А.С.Абрамян

Подписано в печать 2/X-86г. ВФ-05720 Формат 60x84/I6
Офсетная печать.Уч.изд.л,0,5 Тираж 299 экз.Ц.8 к.
Зак.тип.№535 Индекс 3624

Отпечатано в Ереванском физическом институте
Ереван 36, Маркаряна 2