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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ

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INSTABILITY IN WHEELER-DE WITT SUPERSPACE

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ԱՆԿԱՅՈՒՆՈՒԹՅՈՒՆ ՈՒԻԼԵՐ ԴԵ ՎԻՏԻ ԳԵՐՏԱՐԱԾՈՒԹՅՈՒՆՈՒՄ

Քննարկվում է սահմանային /սկզբնական/ պայմանների խնդիրը տիեզերագիտությունում՝ քվանտա-երկրաչափադինամիկական Ուիլեր դե Վիտ-Հոկինգի մոտեցման շրջանակներում: Ցույց է տրված ուղղագծերի /գեոդեզիկ գծերի/ հոսքի անկայունությունը գերտարածությունում և, հետևաբար, Ուիլեր դե Վիտի հավասարման՝ համասեռ տիեզերքներին համապատասխանող լուծումների խիստ կախումը սահմանային պայմաններից:

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Ever greater importance are assumed recently [1-5] ideas on quantizing of geometrodynamics developed by Wheeler and De Witt already in 60th [6,7]. A wave equation for functional of state in superspace was derived by them:

$$\left\{-G_{ijkl}\,\delta^2/\delta\gamma_{ij}\delta\gamma_{kl}+\gamma^{1/2}\left[-{}^{3}R+16\pi T_{00}(\delta/\delta\phi,\phi)\right]\right\}\Psi=0, \quad (1)$$

where Ψ is determined not only by the metric of three dimensional Riemann manifold but also the geometry of superspace. Hawking assumed that the solutions of Wheeler-De Witt equation (1) describe physical state of the Universe if they are determined by a path integral over a compact four-manifold bounded by a compact three-manifold. The attractiveness of this formalism is not only in its great heuristic content but also in predictions being in accordance with observations [5].

A principal problem here is the sensitiveness of solutions of Wheeler-De Witt equation (1) with respect to that boundary (initial) conditions. Below we shall investigate this problem considering the global geometrical properties of superspace and assuming that the searched instability of quantum system is connected with analogous properties of classical system.

Wheeler-De Witt superspace W for given n-dimensional manifold M is the space of all Riemann metrics $\gamma_{ij}(x)$ on M. Below we shall consider only metrics not depended on X corresponding to homogeneous cosmological models. De Witt has considered this spaces before at the case $n=3$ [7]. We shall give also the generalization of De Witt's results for $\dim M = n$, $\dim W = \frac{n(n+1)}{2}$, $n \geq 2$ (see Appendix).

The metric $G^{ijk\ell}$ on the superspace W has a form (cf. [7])

$$G^{ijk\ell} = \frac{1}{2}\gamma^{1/2}\left[\gamma^{ik}\gamma^{j\ell} + \gamma^{i\ell}\gamma^{jk} - 2\gamma^{ij}\gamma^{k\ell}\right], \qquad (2a)$$

where

$$\gamma = \det \gamma_{ij}, \quad \gamma^{ik}\gamma_{kj} = \delta^i_j; \quad i,j,k,\ell = 1,\dots,n.$$

Using the expression

$$G_{ijmn}G^{mnk\ell} = \delta_{ij}^{\ \ k\ell} \equiv \frac{1}{2}\left(\delta^k_i\delta^\ell_j + \delta^\ell_i\delta^k_j\right) \qquad (2b)$$

for $G_{ijk\ell}$ we have

$$G_{ijk\ell} = \frac{1}{2}\gamma^{-1/2}\left[\gamma_{ik}\gamma_{j\ell} + \gamma_{i\ell}\gamma_{jk} - \frac{2}{n-1}\gamma_{ij}\gamma_{k\ell}\right].$$

From (1) one can derive (see Appendix)

$$G^{ijk\ell}d\gamma_{ij}d\gamma_{k\ell} = -d\xi^2 + \frac{n\xi^2}{16(n-1)}\operatorname{tr}\left[\gamma^{-1}\frac{\partial\gamma}{\partial\xi^A}\gamma^{-1}\frac{\partial\gamma}{\partial\xi^B}\right]d\xi^A d\xi^B, \qquad (3)$$

where

$$\xi = 4\left(\frac{n-1}{n}\right)^{1/2}\gamma^{1/4}; \quad \gamma^{-1} \equiv \gamma^{ij}; \quad A,B = 1,\dots,\frac{n(n+1)}{2}-1;$$

the coordinates ξ^A are determined in order the vector $\partial/\partial\xi$ to be orthogonal to $\partial/\partial\xi^A$.

From (3) one can see that G has Lorentzian signature $(-,+,+,\dots,+)$.

The metric on a subspace $\overline{W} = \{\gamma_{ij} \mid \gamma_{ij} \in W, \gamma = 1\}$ induced by the metric G on W has a form

$$\overline{G}_{AB}d\xi^A d\xi^B = \operatorname{tr}\left[\gamma^{-1}\frac{\partial\gamma}{\partial\xi^A}\gamma^{-1}\frac{\partial\gamma}{\partial\xi^B}\right]d\xi^A d\xi^B$$

and Riemannian signature $(+,+,\dots,+)$. The following relations are satisfied for this metric* (see Appendix)

$$\overline{R}_{AB} = -\frac{n}{4}\overline{G}_{AB}, \qquad (4)$$

$$R_{ABCD;E} = 0. \qquad (5)$$

From (4),(5) it follows that $\overline{W}$ with metric $\overline{G}_{AB}$ is a simmetric Einstein space and therefore the geodesics are maximal on $\overline{W}$.

Consider a geodesical flow on $\overline{W}$. We are interested in the stability of that flow respect to initial data, i.e. the variation of the geodesic at small changing of initiel coordinate and velocity. For if one must proceed from Jacobi equation [8,9]

$$\overline{\nabla}_u\overline{\nabla}_u z + \overline{R}(z,u)u = 0, \qquad (6)$$

where $\overline{\nabla}$ denotes covariant derivative, u is the velocity on geodesics $\overline{G}(u,u) = \|u\|^2 = 1$, z is the vector of deviation.

Rewrite Eq.(6) in Fermi coordinates. Let $c(t)$ be a geodesic on $\overline{W}$ manifold, $u(t)$ the velocity of the former. Choose a $(n(n+1)/2-1)$-dimensional orthonormalized basis $\{E_A\}$ on the point $c(0)$ with the last vector coinciding with $u(0)$ and transfer it along $c(t)$. Using this (Fermi) coordinates the Jacobi equation (6) turns to

$$\ddot{z}^a(t) + K^a_b(t)z^b(t) = 0, \qquad (7)$$

where

$$z = z^a E_a, \quad \overline{G}(z,u) = \langle z,u\rangle = 0; \quad a,b = 1,\dots,\frac{n(n+1)}{2}-2; \qquad (8)$$

*Note that there are errors in [7] while deriving these formulae, i.e. in final expression (4) must be $\frac{n}{4}$ instead of $\frac{n}{2}$.

$$K^a_\ell(t) = \langle E^a, \bar{R}(E_\ell, u)u\rangle = \bar{R}_{\ell c d}{}^a u^c u^d ; \qquad (9)$$

$\{E^a\}$ is a dual to $\{E_a\}$ basis; $\langle E^a, E_\ell\rangle = \delta^a_\ell$; t is affine parameter on the geodesic.

Taking into account that

$$\bar{\nabla}\bar{R} = 0,$$

$$\bar{\nabla}_u u = \bar{\nabla}_u E_a = \bar{\nabla}_u E^\ell = 0,$$

we conclude that $K^a_\ell(t)$ does not depend on t :

$$K^a_\ell(t) = K^a_\ell . \qquad (10)$$

In this case the Jacobi equation (7) has a solution

$$z^a_{(i)}(t) = x^a_{(i)} \exp\left[\pm(-\omega_i)^{1/2}, t\right],$$

where $X_{(i)}$ is eigenvector for K^a_ℓ with eigenvalue ω_i :

$$K^a_\ell x^\ell_{(i)} = \omega_i x^a_{(i)} ;$$

i denotes the number of such solutions.

From (4), (8) one has

$$\sum \omega_i = \mathrm{tr}\, K = \bar{R}_{a\ell} u^a u^\ell = -\frac{n}{4} < 0 .$$

We see that for every geodesic there exists such i_0 that $\omega_{i_0} < 0$, which means the existence of at least one non-zero Lyapunov characteristic index. Therefore the geodesic flow is exponentially unstable on $\bar{W}$, i.e. its every geodesic is unstable.

As it is shown in the Appendix at $n=2$ $\bar{W}$ with metric $\frac{1}{2}\bar{G}_{AB}$ is a Lobachevsky space and the geodesic flow there is strongly unstable in the sence that all conditions of Anosov systems besides that of compactness are satisfied.

We see that the geodesic flow is unstable on $\bar{W}$; so far as the geodesics on $\bar{W}$ are the projection of geodesics on W^*,

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*It is always true if the metric on W has a form $ds^2 = \pm d\xi^2 + a^2(\xi)d\sigma^2$, where $d\sigma$ is the metric on $\bar{W}$ and does not depend on ξ .

the latters must be unstable too, if the projection is not a point. Otherwise (i.e. when it is a point) as it is not difficult to show the hypersurface of constant ξ is again unstable, but not exponentially: it expands linearly. Note that this last case is analogous to Friedmann model of the expanding universe with $a(t) = t$.

Thus the results of our analysis show the exponential instability of geodesic flow in Wheeler-De Witt superspace, and therefore the strong dependence from initial (boundary) conditions.

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Appendix

The metric γ_{ij} on M we rewrite in a form: $\gamma_{ij} = \gamma^{1/n}\tilde{\gamma}_{ij}$. Coordinates ξ , ξ^A $\left(A = 1,\dots,\frac{n(n+1)}{2} - 1\right)$ are choosed in a way

$$\gamma_{,A} \equiv \frac{\partial\gamma}{\partial\xi^A} = 0 , \quad \xi = 4\left(\frac{n-1}{n}\right)^{1/2}\gamma^{1/4},$$

so that

$$\partial\tilde{\gamma}_{ij}/\partial\xi = 0 .$$

Then one can show that the following relations take place:

$$\frac{\partial\xi}{\partial\gamma_{ij}} = \frac{1}{4}\xi\gamma^{ij} , \qquad G_{ijk\ell}\frac{\partial\xi}{\partial\gamma_{ij}} = -\frac{4}{n}\xi^{-1}\gamma_{k\ell} ,$$

$$\frac{\partial\gamma_{ij}}{\partial\xi} = \frac{4}{n}\xi^{-1}\gamma_{ij} , \qquad \gamma^{ij}\frac{\partial\gamma_{ij}}{\partial\xi^A} = 0 ,$$

whence it follows that

$$G^{ijkl}\frac{\partial\gamma_{ij}}{\partial\xi}\frac{\partial\gamma_{kl}}{\partial\xi}=-1,\qquad G^{ijkl}\frac{\partial\gamma_{ij}}{\partial\xi^A}\frac{\partial\gamma_{kl}}{\partial\xi}=0,$$

$$G^{ijkl}\frac{\partial\gamma_{ij}}{\partial\xi^A}\frac{\partial\gamma_{kl}}{\partial\xi^B}=\gamma^{1/2}\gamma^{ik}\gamma^{jl}\frac{\partial\gamma_{ij}}{\partial\xi^A}\frac{\partial\gamma_{kl}}{\partial\xi^B}.$$

The metric G on M in coordinates ξ, ξ^A can be written now as follows:

$$G=-d\xi^2+\frac{n\xi^2}{16(n-1)}\operatorname{tr}\left(\gamma^{-1}\gamma_{,A}\gamma^{-1}\gamma_{,B}\right)d\xi^A d\xi^B.$$

In order to calculate the curvature for metric G we need the several more relations following from formulae above:

$$\operatorname{tr}\left(\gamma_{,A}\frac{\partial\xi^B}{\partial\gamma}\right)=\delta_A^B,\quad \operatorname{tr}\left(\gamma\frac{\partial\xi^A}{\partial\gamma}\right)=0,\quad \operatorname{tr}\left(\gamma^{-1}\gamma_{,A}\right)=0,$$

$$\frac{\partial\gamma_{ij}}{\partial\xi^A}\frac{\partial\xi^A}{\partial\gamma_{kl}}=\delta_{ij}^{kl}-\frac{1}{n}\gamma_{ij}\gamma^{kl}.$$

For arbitrary $n\times n$ matrices M, N

$$\operatorname{tr}(\gamma_{,A}M)\operatorname{tr}\left(N\frac{\partial\xi^A}{\partial\gamma}\right)=\frac{1}{2}\operatorname{tr}(MN+MN^T)-\frac{1}{n}\operatorname{tr}(\gamma M)\operatorname{tr}(\gamma^{-1}N),$$

where T denotes transponied matrix and

$$\operatorname{tr}(\gamma_{,AB}M)\operatorname{tr}\left(N\frac{\partial\xi^A}{\partial\gamma}\right)+\operatorname{tr}(\gamma_{,A}M)\operatorname{tr}\left(N\left(\frac{\partial\xi}{\partial\gamma},_B\right)\right)=\frac{1}{n}\operatorname{tr}(\gamma^{-1}\gamma_{,B}\gamma^{-1}N)-$$
$$-\frac{1}{n}\operatorname{tr}(\gamma_{,B}M)\operatorname{tr}(\gamma^{-1}N),$$

$$\operatorname{tr}\left(\gamma_{,A}M\frac{\partial\xi^A}{\partial\gamma}N\right)=\frac{1}{2}\operatorname{tr}(MN^T)+\frac{1}{2}\operatorname{tr}M\cdot\operatorname{tr}N-\frac{1}{n}\operatorname{tr}(\gamma M\gamma^{-1}N).$$

Using this formulae we obtain

$$\bar G^{AB}=\operatorname{tr}\left(\gamma\frac{\partial\xi^A}{\partial\gamma}\gamma\frac{\partial\xi^B}{\partial\gamma}\right),\quad\text{where}\quad \bar G_{AB}\bar G^{CB}=\delta_A^B,$$

$$\bar\Gamma_{ABC}\equiv\frac{1}{2}(\bar G_{AC,B}+\bar G_{BC,A}-\bar G_{AB,C})=\frac{1}{2}\operatorname{tr}\left[\gamma_{,C}\gamma^{-1}(-\gamma_{,A}\gamma^{-1}\gamma_{,B}-\gamma_{,B}\gamma^{-1}\gamma_{,A}+2\gamma_{,AB})\gamma^{-1}\right].$$

$$\bar\Gamma_{AB}^{\ \ C}\equiv\bar G^{CD}\bar\Gamma_{ABD}=\frac{1}{2}\operatorname{tr}\left[\left(\gamma_{,A}\gamma^{-1}\gamma_{,B}-\gamma_{,B}\gamma^{-1}\gamma_{,A}+2\gamma_{,AB}\right)\frac{\partial\xi^C}{\partial\gamma}\right],$$

$$\bar R_{ABC}^{\ \ \ \ D}\equiv\bar\Gamma_{BC\ ,A}^{\ \ D}-\bar\Gamma_{AC\ ,B}^{\ \ D}+\bar\Gamma_{BC}^{\ \ E}\bar\Gamma_{AE}^{\ \ D}-\bar\Gamma_{AC}^{\ \ E}\bar\Gamma_{BE}^{\ \ D}=$$

$$=\frac{1}{4}\operatorname{tr}\left[\left(\gamma_{,C}\gamma^{-1}(\gamma_{,A}\gamma^{-1}\gamma_{,B}-\gamma_{,B}\gamma^{-1}\gamma_{,A})-(\gamma_{,A}\gamma^{-1}\gamma_{,B}-\gamma_{,B}\gamma^{-1}\gamma_{,A})\gamma^{-1}\gamma_{,C}\right)\frac{\partial\xi^D}{\partial\gamma}\right].$$

$$\bar R_{ABCD}\equiv\bar R_{ABC}^{\ \ \ \ E}\bar G_{ED}=\frac{1}{4}\operatorname{tr}\left[(\gamma^{-1}\gamma_{,D}\gamma^{-1}\gamma_{,C}-\gamma^{-1}\gamma_{,C}\gamma^{-1}\gamma_{,D})(\gamma^{-1}\gamma_{,A}\gamma^{-1}\gamma_{,B}-\gamma^{-1}\gamma_{,B}\gamma^{-1}\gamma_{,A})\right].$$

$$\bar R_{AB}\equiv\bar R_{CAB}^{\ \ \ \ C}=-\frac{n}{4}\bar G_{AB},\qquad \bar R\equiv\bar R_{AB}\bar G^{AB}=-\frac{n^2}{4},$$

$$R_{ABCD;E}\equiv\bar R_{ABCD,E}-\bar R_{FBCD}\bar\Gamma_{AE}^{\ \ F}-\bar R_{AFCD}\bar\Gamma_{BE}^{\ \ F}-\bar R_{ABFD}\bar\Gamma_{CE}^{\ \ F}-\bar R_{ABCF}\bar\Gamma_{DE}^{\ \ F}=0.$$

When $n=2$ we have

$$\bar R_{AB}=-\frac{1}{2}\bar G_{AB},$$

i.e. the space $\bar W$ is conform to that of Lobachevsky. It is interesting that in this case the explicit form of metric $\bar G_{AB}$ is possible to find if using the following coordinates

$$\gamma_{ij}=\exp[r\phi]_{ij},$$

where

$$\phi=\begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix},\quad r\geq 0,\quad 0\leq\varphi\leq 2\pi.$$

It is easy to see that

$$\exp[r\phi]=\operatorname{ch}r+\phi\operatorname{sh}r$$

and

$$G_{rr}=\operatorname{tr}(\gamma^{-1}\gamma_{,r}\gamma^{-1}\gamma_{,r})=2,$$
$$G_{\varphi\varphi}=\operatorname{tr}(\gamma^{-1}\gamma_{,\varphi}\gamma^{-1}\gamma_{,\varphi})=2\operatorname{sh}^2 r,$$
$$G_{r\varphi}=\operatorname{tr}(\gamma^{-1}\gamma_{,r}\gamma^{-1}\gamma_{,\varphi})=0,$$

Whence

$$\bar{G} = 2\left[dr^2 + sh^2 r \, d\phi^2\right].$$

For this metric $\bar{R} = -1$, just as it was expected.

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