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V.G. GURZADYAN, A.A. KOCHARYAN

THE ARNOWITT-DESER-MISNER PRINCIPLE AND THE INFLATIONARY UNIVERSE

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By means of ADM principle an approach to the investigation of Hamiltonian systems in superspace, i.e. on a manifold with pseudo-Riemannian metric is developed. For considered inflationary solutions describing the dynamics of the Universe with massive scalar field the laws of stability are found for different potentials.

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1. Introduction

The problem of stability of cosmological solutions being principal for investigation of evolution of the Universe, is a stability problem in Wheeler-De Witt superspace with pseudo-Riemannian (Lorentzian) metric [1]. Therefore the question of investigation of the stability of Hamiltonian systems via the study of global geometrical properties of those spaces (another example is the real space-time possessing Lorentzian signature as well) is arised. The approach used by us is based on the consideration of close geosedics' behaviour onto distinct metrical manifold being the configurational space of a Hamiltonian system [2]. The use of the method of geodesics is principal for our problem: it differs from simple analyses of a perturbated solution of differential equation so far as allows to follow the true deviation of close trajectories and not only by their distinct points. Thus, it is not difficult to see that in our problem two points on close geodesics can run away from each other, while the geodesics themselves can not deviate (cf. [3]).

The first step on this way is the representation of the model of the Universe filled by scalar field as a Hamiltonian system with given Hamiltonian. One can do this using the Hilbert variationary principle in modified form of Arnowitt-Deser--Misner (ADM) ([4] , see also [5,6]).

As an important example we have considered in detail the behaviour of inflationary solutions, hoped to solve several key problems of Cosmology [7-10] . Our analysis being in some sense the continuation of papers [11] enables to find also the laws of stability (decreasing of fluctuations) for different scalar fields.

2. The Arnowitt-Deser-Misner Principle

It is known that Einstein equations can be derived from the principle of least action

$$\delta (I_g + I_m) = 0 ,$$

where I_m and I_g are the actions of the matter and gravitation field, respectively [4,5] :

$$I_g = \frac{1}{16\pi G} \left\{ \int_M d^4x\, g^{1/2} R + 2 \int_{\partial M} d^3x\, h^{1/2} K \right\} . \qquad (1)$$

In eq.(1) R is the scalar curvature, K is the trace of the second fundamental form of the edge, h is the metric induced on it, ∂M is the compact space-like manifold, $g = |\det g_{a\ell}|$ (a system of units when $\hbar = c = 1$ is used). While considering the geometry dynamics in Arnowitt-Deser-Misner formulation [4] , the 3-geometries of the initial and final space-like hypersurfaces are considered to be given. The interval of action is extremal to the space-time choice between these hypersurfaces. Thus, the space-time is stratified into a one-parametric family of space-like hypersurfaces (compact ones, as mentioned above). Then one can choose the surface S with coordinates $X^i (i = 1,2,3)$ in accord with condition of t = 0. The metric of the space-time (3+1) manifold can be written as:

$$ds^2 = g_{a\ell}\, dx^a dx^\ell = -(N^2 - X_i X^i) dt^2 + 2X_i dx^i dt + h_{ij} dx^i dx^j , \qquad (2)$$

where N is the so-called lapse function, allowing to calculate the internal time between two surfaces of constant t ; X_i is the shift vector describing the divergence of a curve with constants from the normal one to S.

In this case the ADM gravitational action will look like

$$I = \frac{1}{16\pi G} \int dt \int d^3x \left[\pi^{ij} \frac{\partial h_{ij}}{\partial t} - N\mathcal{H} - X_i \mathcal{P}^i \right] + \text{s.t.} , \qquad (3)$$

where

$$\mathcal{H} \equiv G_{ijk\ell}\, \pi^{ij} \pi^{k\ell} - h^{1/2}\, {}^3R ,$$
$$G_{ijk\ell} \equiv \frac{1}{2} h^{-1/2} (h_{ik} h_{j\ell} + h_{i\ell} h_{kj} - h_{ij} h_{k\ell}), \qquad (4)$$
$$\mathcal{P} \equiv -2\pi^{ij}{}_{|j} , \quad h \equiv \det h_{ij} ,$$

R is the scalar curvature of the manifold S; the symbol " | " indicates the covariant differentiation with respect to the metric h_{ij} . The expression (3) is varied for independent h_{ij}, π^{ij} , N, X_i .

Below we shall consider locally isotropic and homogeneous cosmological models with scalar fields, i.e. when the metric on the compact 3-manifold S depends on the single parameter a :

$$h_{ij} = \sigma^2 a^2 \tilde{h}_{ij} ,$$

where

$$\sigma^2 = \frac{4\pi G}{3}\left[\int_S d^3x \, \tilde{h}^{1/2}\right]^{-1} , \quad \tilde{h} \equiv \det \tilde{h}_{ij} .$$

Curvature of the metric $\tilde{h}_i$ is equal to (see also Ref. [16])

$$^3R_{ijk\ell} = k \left(\tilde{h}_{ik} \tilde{h}_{j\ell} - \tilde{h}_{i\ell} h_{jk} \right) \qquad (5)$$

with k = +1 (when S is a 3-sphere and 3-sphere factorized over a discrete group), k = 0* (S is a 3-torus or another two-dimensional space), k = -1 (S is a 3-hyperbolic space factorized over a discrete group).

Close to S the 4-metric has the form

$$ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j . \qquad (6)$$

Taking into consideration that the scalar field Lagrangian

$$L_\phi = -g^{-1/2} \left[\phi_{,a} \phi_{,\ell} g^{a\ell} + V(\phi) \right]$$

the action

$$I = I_g + I_\phi$$

can be written in the form (see Ref. [11])

$$I = \int P_\alpha d\alpha + P_\chi d\chi - N \mathcal{H}_{ADM} dt , \qquad (7)$$

where

---

* At k = 0 we require the condition $\int_S d^3x \, \tilde{h}^{1/2} = 1$ to be satisfied.

$$\mathcal{H}_{ADM} = \frac{e^{-3\alpha}}{2} \left[-P_\alpha^2 + P_\chi^2 \right] + e^{3\alpha} \left[U(\chi) - \frac{k}{2} e^{-2\alpha} \right] , \qquad (8)$$

$$\alpha = \ln a , \ \chi = \sigma \phi , \ U(\chi) = \frac{4\pi G \sigma^2}{3} V(\phi) ,$$

and the variation is performed for $\alpha, \chi, P_\alpha, P_\chi$ and N. Variation over N leads to

$$\mathcal{H}_{ADM} = 0 . \qquad (9)$$

3. Reduction of Hamiltonian System to Geodesical Flow on Pseudo-Riemannian Space

Thus, we came to a Hamiltonian system with Hamiltonian

$$\mathcal{H} = \frac{1}{2} g^{a\ell} P_a P^\ell + V(x) \qquad (10)$$

and coupling equation $\mathcal{H} = 0$.

The motion of this system takes place over the extrema of the action

$$I = \int_\gamma P_a dx^a - \mathcal{H}(p, x) d\tau \qquad (11)$$

under the condition of $\mathcal{H} = 0$.

Let us introduce some notations used below

$$v^a \equiv \dot{x}^a = \frac{dx^a}{d\tau} = P_a g^{a\ell} , \quad u^a = \frac{dx^a}{ds} \qquad (12)$$

$$P_a = g_{a\ell} v^\ell ,$$

where

$$g_{a\ell} g^{\ell c} = \delta_a^c$$

(up to now S is an arbitrary parameter).

Let us also notate regions on the hypersurface:

$$W^+\{x\,|V(x)>0\}\quad , \quad W^-\{x\,|V(x)<0\}$$

and also

$$\mathrm{ext}\, I = \gamma_{ext} \;,$$

where

$$\delta I(\gamma_{ext}) = 0 .$$

Then one has

$$\mathrm{ext}\, I\Big|_{\mathcal{H}=0} = \mathrm{ext}\int P_a dx^a\Big|_{\substack{\mathcal{H}=0\\ P_a = g_{a\ell}v^\ell}} = \mathrm{ext}\int g_{a\ell} v^a v^\ell d\tau\Big|_{\mathcal{H}(g_{a\ell}v^\ell,\, x)=0} \;, \qquad (13)$$

where

$$\mathcal{H}(g_{a\ell} v^a v^\ell, x) = \frac{1}{2} g_{a\ell} v^a v^\ell + V(x) = 0 \qquad (14)$$

Assuming that g is a Riemann metric in the region of W^- , one can write

$$g_{a\ell} v^a v^\ell = -2V > 0$$
$$d\tau = \left(\frac{g_{a\ell} u^a u^\ell}{-2V}\right)^{1/2} ds \qquad (15)$$

and using (13)

$$\mathrm{ext}\, I\Big|_{\mathcal{H}=0} = \mathrm{ext}\int\left[-2V\left(\frac{g_{a\ell}u^a u^\ell}{-2V}\right)^{1/2}\right]ds = \mathrm{ext}\sqrt{2}\int(-V g_{a\ell} u^a u^\ell)^{1/2} ds =$$

$$= \mathrm{ext}\int (G_{a\ell} u^a u^\ell)^{1/2} ds \;,$$

where $G_{a\ell} = -V g_{a\ell}$ is a Riemann metric too.

Choose S in order to satisfy [2]

$$\|u\|^2 = G_{a\ell} u^a u^\ell = 1 \;,$$
$$G_{a\ell} u^a u^\ell = -V g_{a\ell} v^a v^\ell \left(\frac{d\tau}{ds}\right)^2 = 2V^2\left(\frac{d\tau}{ds}\right)^2 , \qquad (16)$$

whence

$$ds = \sqrt{2}(-V)\, d\tau \;; \qquad (17)$$

extrema of the action I

$$\mathrm{ext}\int (G_{a\ell} u^a u^\ell)^{1/2} ds = \mathrm{ext}\,\frac{1}{2}\int G_{a\ell} u^a u^\ell ds \;,$$

are reduced to geodesics in the region of W^-:

$$\mathcal{H} = \frac{1}{2} g^{a\ell} P_a P_\ell + V \leftrightarrow \{G_{a\ell} = -V g_{a\ell},\, ds = \sqrt{2}(-V)d\tau,\, \|u\|^2 = 1\}. \quad (18)$$

As to the region W^+, it is obvious that the classical system can not penetrate into it.

Let now g be a pseudo-Riemann metric with signature (-, -, ..+, +, .+). Then at $x \in W^-$ (cf. 2)

$$g_{a\ell} v^a v^\ell = -2V > 0,$$

one again comes to the eq.(18).

If $x \in W^+$

$$g_{a\ell} v^a v^\ell = -2V < 0$$

and one obtains

$$\mathcal{H} \leftrightarrow \{-U \tilde{g}_{a\ell} \;,\; \sqrt{2}(-U)\,d\tau \;,\; 1\} \;, \qquad (19)$$

At V = U - E the obtained expression results from Maupertuis principle [2] .

where $\tilde{G}_{ab} = -U\tilde{g}_{ab} = -Vg_{ab}$ with signature (+, ..-, -, ..), and the following notations are introduced

$$\tilde{g}_{ab} = -g_{ab}\ , \quad U = -V\ , \quad \tilde{\mathcal{H}} = -\mathcal{H} = 0\ ,$$

$$(\tilde{g}_{ab} v^a v^b = -2U > 0\ , \quad x \in W^-(U) = W^+(V) = W^+).$$

Turning now to the metric, with previous signature (-, .-+, ,+)

$$\tilde{G}_{ab} \rightarrow G_{ab} = -\tilde{G}_{ab}\ ,$$

one has

$$\mathcal{H} \leftrightarrow \{V g_{ab}, \sqrt{2}\, V\, d\tau, -1\}. \tag{20}$$

Expressions (18) and (29) can be united as follows:

$$\mathcal{H} \leftrightarrow \{|V| g_{ab}, \sqrt{2}\, |V|\, d\tau, -\mathrm{sgn}\, V\}. \tag{21}$$

Thus, the Hamiltonian system is represented as a geodesical flow on pseudo-Riemann manifold. In order to study the stability of this flow, one must proceed from Jacobi equation ([2], for the case given see [12-13]):

$$\frac{d^2 z^i}{ds^2} + K^i_j z^j = 0, \tag{22}$$

where the two-dimensional curvature K^i_j is defined as

$$K^i_j = \langle E^i, R(u, E_j)u \rangle \tag{23}$$

(E^i is Fermi basis, $Z = Z^i E^i$ is the deviation vector of geodesics).

Before solving the eq.(22), one must convert the parameter S into the time τ. In view of

$$ds = \sqrt{2}\, |V|\, d\tau \tag{24}$$

one can obtain

$$\frac{d}{ds^2} = \frac{1}{2V^2}\left[\frac{d^2}{d\tau^2} - \left(\frac{d}{d\tau}\ln|V|\right)\frac{d}{d\tau}\right].$$

Then the Jacobi equation will read

$$\frac{d^2 z^i}{d\tau^2} + \gamma(z)\frac{dz^i}{ds} + \omega^i_j(z) = 0\ , \tag{25}$$

where

$$\gamma(\tau) \equiv -\frac{d}{d\tau}\ln|V|\ ,$$

$$\omega^i_j \equiv 2V^2 K^i_j\ , \quad i, j = 1, \ldots, k-1.$$

Substituting the variables

$$z^i = A Y^i\ ,$$

the eq.(25) can be rewritten in the form

$$\ddot{Y}^i + 2\left(\frac{\dot{A}}{A} + \gamma\right)\dot{Y}^i + \left[\left(\frac{\ddot{A}}{A} + \gamma\frac{\dot{A}}{A}\right)\delta^i_j + \omega^i_j\right]Y^j = 0\ , \tag{26}$$

or

$$\ddot{Y}^i + \left[\omega^i_j - \left(\frac{1}{2}\dot{\gamma} + \frac{\gamma^2}{4}\right)\delta^i_j\right]Y^j = 0\ , \tag{27}$$

if

$$\frac{\dot{A}}{A} = -\frac{1}{2}\gamma\ , \quad A = |V|^{1/2}.$$

In particular case of m = 2 we have

$$K^1_1 = \langle E^1, R(u, E_1)u \rangle = \langle E^1, K[\|u\|^2 E_1 - \langle u, E_1 \rangle u] \rangle = K\|u\|^2 \tag{28}$$

and equation

$$\ddot{Y} + \left[\omega - \left(\frac{1}{2}\dot{\gamma} + \frac{\gamma^2}{4}\right)\right]Y = 0\ , \tag{29}$$

where

$$\gamma = -\frac{d}{d\tau}\ln|V|\ , \quad \omega = 2V^2 K \|u\|^2.$$

In accord with above expressions for the action I, we have

$$\mathcal{H} \leftrightarrow \left\{ G\eta_{ab},\ \sqrt{2}\,|v|\,d\tau,\ -\operatorname{sgn} V \right\}, \tag{30}$$

where

$$N\,dt = d\tau, \quad G = e^{6\alpha}\left|U - \frac{k}{2}e^{-2\alpha}\right|,$$
$$|v| = e^{3\alpha}\left|U - \frac{k}{2}e^{-2\alpha}\right|. \tag{31}$$

When k = 0 these formulae are still more simplified

$$G = e^{6\alpha}|U|, \quad |v| \doteq e^{3\alpha}|U|,$$
$$\gamma(\tau) = -3\frac{d\alpha}{d\tau} - \frac{1}{U}\frac{dU}{d\tau} = -3H - \frac{1}{U}\frac{dU}{d\tau},$$
$$K(\tau) = -\frac{1}{2}\frac{\Box \ln G}{G} = -\frac{1}{2}\frac{\ln\left(e^{6\alpha}|U|\right)}{e^{6\alpha}|U|} = -\frac{1}{2}\frac{\Box \ln|v|}{e^{6\alpha}|U|},$$
$$\omega(\tau) = 2e^{6\alpha}U^2\left[-\frac{1}{2}\frac{\Box \ln|U|}{e^{6\alpha}|U|}\right]\cdot(-\operatorname{sgn} U) = U\Box\ln|U| = \tag{32}$$
$$= U'' - \frac{U'^2}{U}; \quad \left(\Box \equiv -\frac{\partial}{\partial\alpha^2} + \frac{\partial^2}{\partial\chi^2},\ U' \equiv \frac{dU}{dx}\right).$$

4. The Inflationary Stage

Now let us turn to the investigation of the stability of "inflationary" solutions. Assuming that k = 0, we proceed from the fact, that this stage quickly flattens the Universe even proceeded from the initial value k ≠ 0.

As it is shown in Ref.[11] the inflationary stage can realize at rather general initial conditions. The following conditions are satisfied on this stage([9, 14])

$$\dot H \ll H^2, \quad \dot x^2 \ll U, \quad \left|\frac{\dot x}{x}\right| \ll H$$
$$x \gg 1 \text{ at } \alpha \to -\infty \tag{33}$$

According to [9] a large variety of scalar potentials (polynomial, etc.) lead to the inflationary stage.

Let us consider the scalar potential in a general form

$$U = \frac{\lambda\chi^n}{n}. \tag{34}$$

Then, using the eqs.(32) one has

$$\gamma = -3H - n\frac{\dot\chi}{\chi} \sim -3H,$$
$$\omega = -\frac{nU}{\chi^2} \tag{35}$$

whence

$$\dot\gamma = -3\dot H,$$
$$\frac{1}{2}\dot\gamma + \frac{\gamma^2}{4} \simeq \frac{9}{4}H^2; \tag{36}$$

the Jacobi equation reading:

$$\ddot Y - \left[-\frac{nU}{\chi^2} + \left(\frac{3}{2}H\right)^2\right]Y = 0. \tag{37}$$

From Einstein equation

$$\dot H + 3H^2 = 6U \tag{38}$$

and the condition $\dot H \ll H^2$ for the Hubble constant, one has

$$H^2 \sim 2U. \tag{39}$$

If

$$\chi \gg \left(\frac{2n}{3}\right)^{1/2},$$

then

$$\frac{nU}{\chi^2} \sim \frac{nH^2}{2\chi^2} \ll \frac{9}{4}H^2. \tag{40}$$

Consequently, eq.(37) will reduce to

$$\ddot{Y} - \left(\frac{3}{2}\dot{\alpha}\right)^2 Y = 0 \qquad (41)$$

and at $\alpha \to -\infty$, has a solution

$$Y \simeq \mathrm{const}\, e^{\pm\frac{3}{2}\alpha}$$

or

$$Z_\pm \simeq e^{\frac{3}{2}\alpha} |U|^{1/2} \mathrm{const}\, e^{\pm\frac{3}{2}\alpha};$$

$$Z_+ \simeq \mathrm{const}\, |U|^{1/2}, \quad Z_- \simeq \mathrm{const}\, |U|^{1/2} e^{3\alpha}. \qquad (42)$$

Finally, for Z one obtains (assuming that $Z_- << Z_+$)

$$Z = C_+ Z_+ + C_- Z_- \simeq \mathrm{const}\, |U|^{1/2}, \qquad (43)$$

whence

$$\dot{Z} \simeq \frac{1}{2} \frac{U'}{U^{1/2}} \dot{\chi}. \qquad (44)$$

Using the relation [15]

$$\dot{\chi} \simeq \mathrm{const}\, \frac{U}{U^{1/2}},$$

one has from (44) (for more details see Ref. [12])

$$\delta^2 = Z^2 + \dot{Z}^2 \simeq \mathrm{const}\, U + \mathrm{const}\, \frac{U'^4}{U^2} \simeq \mathrm{const}\, \chi^n + \mathrm{const}\, \chi^{2n-4}. \qquad (45)$$

The scalar field potential decreases during expansion ($H > 0$), i.e. χ decreases at any $n \geqslant 2$, and therefore, one can say that the considered inflationary solutions are stable by Lyapunov. Moreover, the formula (45) allows one to find the law of decreasing of perturbations for each particular potential. I.e. in the frequently discussed simplest case $n = 2$, i.e. at

$$U = \frac{\lambda \chi^2}{2}, \qquad (46)$$

one has

$$Z \simeq \chi \simeq -t, \quad \dot{Z} \simeq -1,$$
$$\delta \left| Z^2 + \dot{Z}^2 \right|^{1/2} \simeq -t. \qquad (47)$$

In this case the inflationary solutions are linearly stable by Lyapunov's criterion.

At $n = 4$, as it is easily seen, perturbations decrease exponentially; the larger n, the more stable are the solutions.

Thus, an approach to the investigation of Hamiltonian systems on pseudo-Riemannian manifolds is developed. By means of ADM principle the Wheeler-De Witt superspace is investigated. The exponential-inflationary solutions describing the dynamics of the Universe filled with massive scalar field, are analyzed in detail. The laws of decreasing of fluctuations are found for fields with different potentials. Together with earlier results [8,11] it convinces the ideas on inflationary stage of the Universe evolution.

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ПРИНЦИП АРНОВИТТА-ДЕЗЕРА-МИЗНЕРА И РАЗДУВАЮЩАЯСЯ ВСЕЛЕННАЯ

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