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INFINITE-DIMENSIONAL GAUGE ISING MODEL

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ЕРЕВАН — 1987

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Ն.Գ. ՏԵՐ-ՀԱՐՈՒԹՅՈՒՆՅԱՆ-ՍԱՎՎԻԴԻ

ԻԶԻՆԳԻ ԱՆՎԵՐՋ ՉԱՓՈՂԱԿԱՆՈՒԹՅԱՆ ՏՐԱՄԱՉԱՓԱԿԱՆ ՄՈԴԵԼԸ

Ձևակերպված է իզինգի անվերջ չափողականության տրամաչափական մոդելը ցանցի վրա: Արտածված են ռեկուրենտ առնչություններ, որոնց օգնությամբ հետազոտված են մոդելի կրիտիկական հատկությունները: 1-ին սեռի փուլային անցումների համար գտնված են կրիտիկական ջերմաստիճանը և թաքնված ջերմությունը: Ապացուցված է, որ Փոթսի ,,մաքուր,, Z(Q)-համաչափ մոդելներում, մեծ չափողականությունների դեպքում բացակայում են 2-րդ սեռի փուլային անցումները:

Երևանի Ֆիզիկայի ինստիտուտ

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Н.С.АНАНИКЯН, А.З.АХЕЯН,
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БЕСКОНЕЧНОМЕРНАЯ КАЛИБРОВОЧНАЯ МОДЕЛЬ ИЗИНГА

Сформулирована бесконечномерная калибровочная модель Изинга на решетке. Выведены рекуррентные соотношения, при помощи которых исследованы критические свойства модели. Найдены критическая температура и скрытая теплота фазового перехода I рода. Доказано отсутствие фазового перехода II рода в "чистых" $Z(Q)$ -симметричных моделях Поттса при больших размерностях.

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N.G. TER-ARUTUNYAN-SAVVIDY

INFINITE-DIMENSIONAL GAUGE ISING MODEL

An infinite-dimensional gauge Ising model on the lattice is formulated. The recursion relations are derived, using which the critical properties of the model are studied. The critical temperature as well as latent heat of the I-order phase transition are found. The absence of the II-order phase transition in the "pure" Z(Q)-symmetric Potts models at higher dimensions is proved.

Yerevan Physics Institute

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1. Introduction

The properties of Z(Q)-symmetric models are of certain interest for the elementary particle physics. The Z(Q) group, being a centre of the SU(Q) group, plays a crucial role in quark confinement [1-3] . In addition, while models with global Z(Q)-symmetry are equivalent to the random-walk models and determine the quantum properties of point-like objects, the gauge (local) Z(Q)-symmetric models are connected with the random surface theory [4] . Such models have been intensively studying in the recent years because of their relationship to the string quantization [5-7] .

The present work deals with the Z(2)-gauge model (Ising) on inifinite-dimensional lattice. The "pure" gauge Ising model is shown to have the I order phase transition and have no higher-order ones. However, the continuous field theory is known to correspond to lattice models only near the point of the II order phase transition, since just in this region the correlation length is much larger than the lattice space, so all effects connected with the discrete lattice are wiped out.

The same difficulty arises in the Z(Q) globally symmetric (spin) models at $Q > 2$ in higher dimensions. The I order phase transition takes place at zero external field, and the II order one is achieved only by introducing external field [8]. Similarly, the consideration of the gauge-matter interaction in gauge models allows one to obtain, in principle, the II-order phase transition.

In Sect. 2, we formulate the model as well as derive recursion relations. In Sect. 3, we determine the gauge-invariant thermodynamic quantities: free and internal energies, as well as consider the model critical behavior. Analytical and numerical results are presented in Sect. 4. In Conclusion we discuss some extension of the Z(2) gauge model in infinite dimensions.

2. Formulation of the Model and Recursion Relations

The model is determined on a special lattice - 2-dimensional generalization of the Cayley tree (Fig. 1). The lattice is built up by successive glueing of shells. As a 0-th shell we take one plaquette (a primitive two-dimensional square). Glueing γ new plaquettes to each of its links, we form the 1-st shell. The 2-nd shell is constructed by glueing γ plaquettes to each free link of the first shell. Proceeding with this process, we obtain a two-dimensional infinitely branching surface whose Hausdorff dimension is $d_H = \infty$, this being proved in the same way as in the case of the Bethe lattice [9].

The Z(2) lattice gauge model is defined by the action:

$$S = \beta \sum_{p\ell} u_1 u_2 u_3 u_4 \tag{2.1}$$

where u_l are gauge variables taking the values ±1 on the links; $u_1 u_2 u_3 u_4$ is their product round the plaquette; the sum is taken over all plaquettes of the lattice; β is a temperature-reverse quantity, $\beta = J/KT$. Rewrite the action in the form:

$$S = \beta \sum_{p\ell} \delta_{u_1 u_2 u_3 u_4, 1} \tag{2.2}$$

which comes out from the previous one via replacement of β by 2β and level shift. The partition function will be written as:

$$Z = \sum_{\{u\}} \exp\left\{\beta \sum_{p\ell} \delta_{u_1 u_2 u_3 u_4, 1}\right\} \tag{2.3}$$

The boundary conditions are chosen in the simplest way: the variables on al boundary links take the same value, e.g. ±1 (the change of boundary conditions from +1 to -1, as we shall see, is equivalent to gauge transformation and cannot therefore affect the final results).

The advantage of the lattice introduced is that for models formulated on it the exact recursion relations can be derived. Consider the above-defined lattice with n shells. If we cut it in the sites of the central plaquette, it will fall into 4 disconnected parts. Each part, in turn, represents separate branches connected by the common initial link. Respectively, the partition function can be written as

$$Z = \sum \exp\left(\beta \delta_{u_1^\circ u_2^\circ u_3^\circ u_4^\circ, 1}\right) [g_n(u_1^\circ)]^\gamma [g_n(u_2^\circ)]^\gamma [g_n(u_3^\circ)]^\gamma [g_n(u_4^\circ)]^\gamma \tag{2.4}$$

where the first exponent is a contribution of the central plaquette (of the 0-th shell), and $g_n(u_i^\circ)$ is a contribution of every branch starting from the i-th link (i = 1,2,3,4, index n means that a given branch has n shells). Comparing this expression with the initial one, we find:

$$g_n(u_i^\circ) = \sum_{\{u\}} \exp\left\{\beta \delta_{u_i^\circ u_2^1 u_3^1 u_4^1, 1} + \beta \sum_{p\ell} \delta_{u_1 u_2 u_3 u_4, 1}\right\} \tag{2.5}$$

where summation goes over all variables of the branch, except u°. Each branch, in turn, can be cut in sites of the initial plaquette, and it will fall into one link and 3 disconnected parts. Each part, again, is γ separate branches connected only by the link in the base; however new branches contain already n-1 shells; therefore the expression for $g_n(u)$ can be rewritten in the form:

$$g_n(u^{\circ}) = \sum_{u_i^1} \exp\left(\beta\delta_{u_1^{\circ}u_2^{\circ}u_3^{\circ}u_4^{\circ},1}\right)\left[g_{n-1}(u_2^{\circ})\right]^{\gamma}\left[g_{n-1}(u_3^{\circ})\right]^{\gamma}\left[g_{n-1}(u_4^{\circ})\right]^{\gamma} \tag{2.6}$$

Let us introduce a new variable:

$$x_n \equiv \frac{g_n(-1)}{g_n(+1)} \tag{2.7}$$

Making summation in g_n over u_i^1, we'll obtain for x_n the recursion relations:

$$x_n = y(x_{n-1}), \qquad y(x) = \frac{zx^{3\gamma} + 3x^{2\gamma} + 3zx^{\gamma} + 1}{x^{3\gamma} + 3zx^{2\gamma} + 3x^{\gamma} + z} \tag{2.8}$$

where we introduced the notation $z = e^{\beta}$. The model thermodynamical parameters are expressed, as we'll see, through x_n, and the recursion relations enable us to study clearly the critical behavior.

3. Thermodynamical and Critical Properties of the Model

The above-introduced quantity x is not gauge-invariant: at some gauge transformation, $g_n(1)$ turns to $g_n(-1)$, and vice versa, i.e. x turns to $\frac{1}{x}$. A simplest invariant quantity is a product of variables ($u_1 u_2 u_3 u_4$) along the contour of one plaquette. This quantity averaged over all states we denote by S:

$$S = \langle \delta_{u_1 u_2 u_3 u_4, 1} \rangle = \frac{\sum\limits_{\{u\}} \delta_{u_1 u_2 u_3 u_4, 1} \exp\{\beta \sum\limits_{p\ell} \delta_{u_1 u_2 u_3 u_4, 1}\}}{\sum\limits_{\{u\}} \exp\{\beta \sum\limits_{p\ell} \delta_{u_1 u_2 u_3 u_4, 1}\}} \quad (3.1)$$

The right-hand side being summed over the central plaquette variables u_i°, with account of (2.5) we obtain the expression for S through x:

$$S = \frac{x^{4\gamma} + 6x^{2\gamma} + 1}{x^{4\gamma} + 6x^{2\gamma} + 1 + \frac{4}{z}(x^{3\gamma} + x^{\gamma})} \quad (3.2)$$

To clarify the physical meaning of quantity S, recall the definition of free energy:

$$-\beta F = \ln Z \quad (3.3)$$

whence

$$-\frac{\partial}{\partial \beta}(\beta F) = \frac{1}{Z}\frac{\partial Z}{\partial \beta} = \sum_{p\ell} \langle \delta_{u_1 u_2 u_3 u_4, 1} \rangle \quad (3.4)$$

Introduce the quantity f - average free energy per one plaquette. Then, from (3.4) we have:

$$-\frac{\partial}{\partial \beta}(\beta f) = \langle \delta_{u_1 u_2 u_3 u_4, 1} \rangle = S \quad (3.5)$$

So, quantity S is internal energy of one plaquette. Free energy is found by integrating (3.5) over β:

$$-\beta f(\beta) = \int_0^{\beta} S(\beta')\, d\beta' + C \quad (3.6)$$

Now return back to recursion relation (2.8). In the limit $n \to \infty$ the sequence $\{x_n\}$ reduces to the stable solution of the equation:

$$y(x, z) = x \quad (3.7)$$

if the latter exists. Here two cases are possible:

- if at any value of parameter z (at any temperature) Eq. (3.7) has a single solution, then the system has no phase transition.

- if there exists a bifurcation point z_c in the system (3.7), then the phase transition is possible. In this case, if the change of solutions is step-wise ($x_2 \neq x_1$), then the transition will be of the 1-st order; if the bifurcation is continuous ($x_2 = x_1$ at z_c point), then it means a phase transition of higher order (e.g. of the 2-nd order).

In the considered model, the second case takes place: beginning with $z = z_c$, Eq. (3.7) besides $x_1 = 1$ has a new solution, $x_2(z) \neq 1$. One can easily see that precisely the last solution is physical, i.e. a lesser value of free energy corresponds to it. Indeed, the function $S(x, z)$ (3.2) has a minimum at $x = 1$, and thus the free energy determined from (3.6) for $x_1 = 1$ is larger than for $x_2(z) \neq 1$.

Hence at high temperatures $T > T_c$ $(z < z_c)$, Eq. (3.7) has a single solution $x = 1$. The internal energy respectively has the form:

$$S^{(1)} = \frac{z}{z+1} \tag{3.8}$$

and free energy is defined by integral:

$$f(z) = -\frac{1}{\lg z} \int_1^z \frac{dz}{z'+1} + C \tag{3.9}$$

At $T \geqslant T_c$ $(z \leqslant z_c)$, some $x_2(z) \neq 1$ becomes a physical solution. Correspondingly, the internal energy $S^{(2)}$ is now determined by expression (3.2), where x should be replaced by a concrete function $x_2(z)$.

The quantity

$$L = S^{(2)}(z_c) - S^{(1)}(z_c) \tag{3.10}$$

is called a latent heat of transition. The fact that L is other than zero means a 1-st order phase transition.

The free energy at $T < T_c$ can be presented in the form:

$$f(z) = -\frac{1}{\ln z}\left\{\int_1^{z_c}\frac{dz'}{z'+1} + \int_{z_c}^{z} S^{(1)}[z', x'(z')]\frac{dz'}{z'}\right\} \tag{3.11}$$

and since the function $z > z_c$ is monotonous at $x = x(z)$, then

$$f(z) = -\frac{1}{\ln z}\left\{\int_1^{z_c}\frac{dz'}{z'+1} - \int_{x(z)}^{x(z_c)} S^{(2)}[z(x'), x']\left(\frac{\partial z}{\partial x'}\right)\frac{dx'}{z(x')}\right\} \tag{3.12}$$

The function $z(x)$ being determined from Eq. (3.7), the free energy integral can be calculated analytically.

4. Analytical Solution of Recursion Relations and Numerical Results

Eq. (3.7) can be solved for z:

$$z(x) = \frac{x^{3\gamma+1} - 3x^{2\gamma} + 3x^{\gamma+1} - 1}{x^{3\gamma} - 3x^{2\gamma+1} + 3x^{\gamma} - x} \tag{4.1}$$

At a critical point x_c the function $z(x)$ has a minimum:

$$\left.\frac{dz}{dx}\right|_{x = x_c} = 0 \tag{4.2}$$

Since all observable quantities do not change under transformation of x to $1/x$ (they are gauge-invariant), it is enough to consider the region of x from "0" to "1". Eq. (4.2) via the variable replacement $x = e^y$ reduces to:

$$\operatorname{sh}\gamma y - 3\gamma\operatorname{sh} y = 0 \tag{4.3}$$

Using the properties of hyperbolic functions, we can obtain exact values of x_c for small γ (from 1 to 7), e.g.:

$$x_c = 3 - \sqrt{8} \qquad \text{at} \qquad \gamma = 2$$
$$x_c = \sqrt{4-\sqrt{15}}\ , \text{at} \qquad \gamma = 3 \quad \text{and so on.} \tag{4.4}$$

For the rest of γ, the equation can be solved with a computer. z_c is found by substituting x_c to (4.1). The latent heat can be found from formula (3.10). Table 1 lists data for various γ. Fig. 2 presents z_c as a function of γ. With increasing γ, T_c increases (z_c decreases), and in the limit $\gamma \to \infty$ T_c turns to infinity ($z_c \to 1$). Fig. 3 shows the γ dependence of latent heat of transition L. With increasing γ the latent heat increases tending to a certain limit. This limit can be found numerically. For that, we expand quantity x_c near 1 restricting to the first order at large γ :

$$x_c \approx 1 - \frac{a}{\gamma} \tag{4.5}$$

Substituting this to Eq. (4.3), we obtain the equation for coefficient a :

$$e^{a} - e^{-a} + 6a = 0 \tag{4.6}$$

Its solution is a = 2.839. Using (4.5), from formula (4.1) we find the behavior of z_c at large γ :

$$z_c \approx 1 + \frac{\alpha}{\gamma} \quad \text{where} \quad \alpha = a\left(\frac{e^{a}+1}{e^{a}-1}\right)^3 \approx 4{,}034 \tag{4.7}$$

and from formula (3.10) the upper limit for latent heat:

$$L \to L^* = \frac{1}{2}\left(\frac{e^{a}-1}{e^{a}+1}\right)^4 \approx 0{,}313$$

These results can be compared to those obtained by the mean field method and 1/d-expansion [10] . For that, we determine the quantity:

$$\beta_c^* = \beta_c \gamma = \gamma \ln z_c$$

In the limit $\gamma \to \infty$ the quantity β_c^* has a finite limit $\beta_c^* = \alpha \approx 4.034$. Here one can observe the 1-st order phase transition, just like in the case of mean field and 1/d-expansion. However the value of β_c^* is somewhat different (with account of the recalculated coefficient 1/2 the result of Ref. 10 is $\beta_c^* = 5.01$), this presumably being due to the choice of lattice.

5. Conclusion

So, in the limit of higher dimensions of a lattice, the "pure" (we mean only gauge variables introduced on links) $Z(2)$-gauge models have the phase transition of the 1-st order only. Analogously we can prove the existence of the 1-st order phase transition in the "pure" Z(Q)-symmetric Potts models as well. A more detailed description of Z(Q)-symmetric models we will publish elsewhere.

As was mentioned in Introduction, the 2-nd order phase transition, necessary to obtain continuous theory, can be attained via the introduction of matter fields into the action:

$$S = S_g + S_{mg} = -\beta \sum_{p\ell} \delta_{\Pi \sigma_{ij},1} - K \sum_{\langle ij \rangle} \delta_{s_i \sigma_{ij} s_j^{-1},1} \qquad (5.1)$$

where S_i are the matter fields determined in the sites. In the partition function, the action (5.1) can be reduced by gauge fixing (matter field "freezing") to the form:

$$S = -\beta \sum_{p\ell} \delta_{\Pi\sigma_{ij},1} - K \sum_{\langle ij\rangle} \delta_{\sigma_{ij},1} \tag{5.2}$$

The latter corresponds to the action of the "pure" gauge model in external field. Recursion relations analogous to (2.9) can readily be derived for this model too. Just like in case with Z(Q) global symmetry [8] , the introduced external field allows us to obtain the 2-nd order phase transition.

So, only the Z(Q)-symmetric gauge-matter interacting models (but neither pure gauge nor pure spin models) have continuous field limit and are of interest for elementary particle physics. Hence we can conclude that some "supersymmetry" is required for the Z(Q) lattice gauge model to correspond to continuous field theory.

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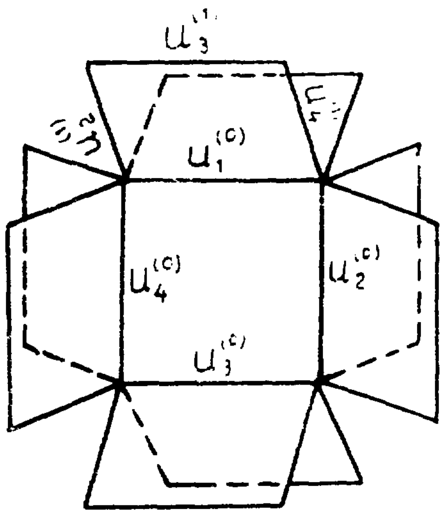

Fig.1

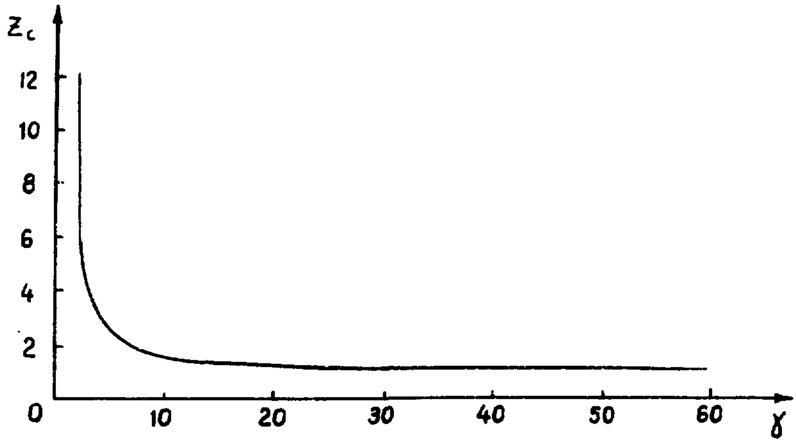

Fig.2

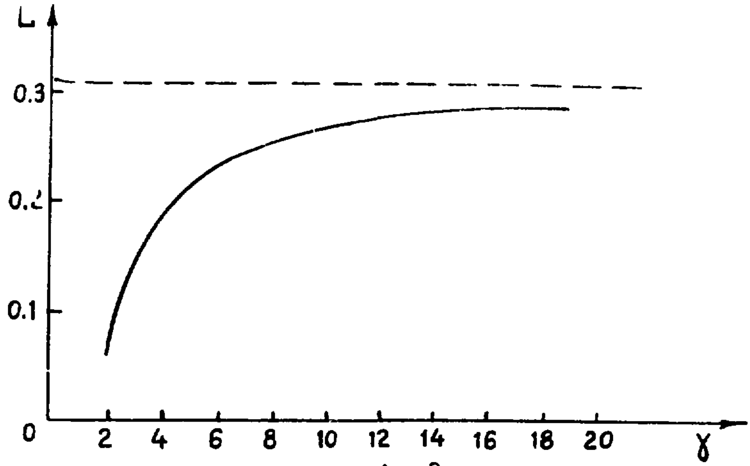

Fig.3

Figure Captions

Fig. 1. Two-dimensional generalization of Cayley tree with $\gamma = 2$ (only the zeroth and first shells are shown).

Fig. 2. Temperature parameter $Z_c = \exp(J/KT_c)$ versus γ , where T_c is temperature of the 1-st order phase transition, and $\gamma + 1$ is the lattice coordinate number.

Fig. 3. The **1-st** order phase transition latent heat L versus γ . With **increasing** γ , L tends to limiting value 0.313.

Table 1

γ	2	3	4	5	9	19	39	120
x_c	0.1717	0.3562	0.4753	0.5573	0.7270	0.8603	0.9290	0.9762
z_c	11.7858	4.2987	2.8691	2.2921	1.5712	1.2368	1.1090	1.0345
L	0.0683	0.1487	0.1929	0.2191	0.2645	0.2929	0.3067	0.3128

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БЕСКОНЕЧНОМЕРНАЯ КАЛИБРОВОЧНАЯ МОДЕЛЬ ИЗИНГА

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