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ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ
YEREVAN PHYSICS INSTITUTE

V.G.GURZADYAN, A.A.KOCHARYAN

WHAT TOPOLOGY COULD THE UNIVERSE BE CREATED WITH?

ЦНИИатоминформ
ЕРЕВАН — 1987

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Հոկինգյան քվանտային տիեզերագիտության շրջանակներում դիտարկված են առաջացող Տիեզերքի տոպոլոգիական և երկրաչափական հատկությունները՝ ոչ զրոյական տիեզերական հաստատունի դեպքում: Հաշված են Տիեզերքի ծնման հավանականությունները տարբեր տոպոլոգիաների /ներառյալ թորի, ոլորակի և հիպերբոլական տարածությունների/ համար. գերաճող Տիեզերքների համար այդ տոպոլոգիաները ունեն հավասար հավանականություն: Որոշակի օրինակի համար գտնված է հետագայում Տիեզերքի տոպոլոգիայի քվանտային փոփոխման հավանականությունը:

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В.Г.ГУРЗАДЯН, А.А.КОЧАРЯН

С КАКОЙ ТОПОЛОГИЕЙ МОЖЕТ РОЖДАТЬСЯ ВСЕЛЕННАЯ?

В рамках хоукинговской квантовой космологии рассматриваются топологические и геометрические свойства рождающейся Вселенной с космологической постоянной. Вычислены вероятности рождения Вселенной с разной топологией (в том числе тора, сферы, гиперболического пространства); для раздувающейся Вселенной эти топологии оказались равновероятными. Для конкретной модели вычислена вероятность квантового изменения топологии в ходе эволюции Вселенной.

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V.G. GURZADYAN, A.A. KOCHARYAN

WHAT TOPOLOGY COULD THE UNIVERSE BE CREATED WITH?

In the framework of Hawking quantum cosmology the topological and geometrical properties of the created Universe wit. cosmological constant are considered. Probabilities for the Universe creation with different topologies (including torus, sphere, hyperbolic space) are calculated. These topologies turned out to be equally probable for the case of inflationary Universe. For a considered toy model the probability for the quantum change of topology during the Universe evolution is calculated.

Yerevan Physics Institute

Yerevan 1987

1. Introduction

The subject of non-trivial topology of the Universe became more actual form the beginning of 60-ies after Wheeler's works on geometrodynamics involving the concepts on foam-like structure of space-time [1]. Due to the recent development of the ideas on quantum origin of the Universe (see, e.g., refs.[2-4]) this question has been somewhat changed: what topology is the creation of the Universe possible with and with what probability [5]? However, as Zeldovich and Starobinsky had noted, difficulties arise already in the formulation of this question, particularly "it is not clear what the probability of creating of a closed Universe is and how to normalize that probability".

Quite encouraging is the discussion of the originating Universe topology in the framework of the quantum cosmology developed by Hawking and coauthors [6-8]. Based on this approach to the determination and interpretation of the wave function of the Universe, several rather important problems concerning the cosmological constant, initial perturbations, inflationary stage, CPT-theorem, etc. had already been considered [9-13].

In the framework of the Euclidean formulation of the functional integral, the conception of probability as well as a natural condition of the wave function normalization is defined. The possibility for direct calculations in quasiclassical approximation, in particular, the calculation of amplitudes of 3-geometries with non-trivial topology is another important advantage (especially in the context of this work) pointed out by Hawking and Hartle [7]. In ref. [7] also the topological aspects of the Universe origination were discussed, including the properties of the 4-manifold, the edge of which is the given 3-manifold (bordant to zero). These aspects were in more details considered by Mkrtchyan [14,15] who succeeded, for particular cases, in obtaining limitations on the properties of the Universe with matter. In this connection it should be noted that according to the cobordism theory any two closed 3-manifolds are cobordant (Rokhlin theorem), i.e. there are no limitations on the topology of the 3-manifold bordant to zero [16]. When the geometry of 3-manifold is adopted, as is well known, it can be smoothly continued onto the 4-manifold (if the condition of paracompactness is satisfied).

In the present paper we shall try to investigate both topological and geometrical properties of the originating Universe in the framework of an approach by Hawking and coauthors. The problem is as follows: in the quasiclassical approximation to estimate the probability for the birth of the Universe with different topologies in the superspace. It appeared, that the consideration of this problem requires the estimation of the solutions of Einstein's equations for the given value of the cosmological constant Λ and spaces without matter:

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} ,$$

in the case with homogeneous, isotropic metrics.

Among these solutions are both compact gravitational instantons (see refs. [17,18]) and complex solutions contributing, as is shown, to the wave function sought for. The latters we call pseudoinstantons.

Having the necessary solutions, we afterwards calculate the Euclidean gravitation action which enters the functional integral of the wave function. Then by the steepest-descents method the wave function integrals for spaces with different topology and hence the relative probabilities of their birth are calculated. Thus, for spaces able to inflate after their birth, these topologies appear to be equally probable.

Accounting the possibility of quantum changing of the topology of the created Universe, the probability amplitudes of some transitions are calculated for a concrete example (at $\Lambda = 0$). It turned out that sphere-torus transition is practically impossible with respect to sphere-sphere transition (i.e. without change of topology).

2. Canonical Quantum Cosmology

According to quantum-geometrodynamical formalism some quantum state of the Universe is described by the wave function $\Psi(S, h_{ij})$ satisfying the Wheeler-De Witt equation in the superspace - on infinite-dimensional space of all Riemann metrics h_{ij} (for the superspace properties see ref. [19]), the square of that wave function determining the probability of the Universe birth on the 3-manifold S with metric h_{ij} (at the absence of matter).

The quantum state of the real Universe as Hawking and co-authers have proposed, is determined by a wave function which is given as

$$\Psi(S, h_{ij}) = \int_C d[g_{\alpha\beta}] \exp(-I[g_{\alpha\beta}]), \qquad (1)$$

where the integral is taken over all four-dimensional compact manifolds M of Euclidean metrics $g_{\alpha\beta}$ inducing the metric h_{ij} on the boundary $\partial M = S$.

The Euclidean gravitational action has the form

$$I[g_{\alpha\beta}] = -\frac{1}{16\pi \ell^2} \int_M d^4x\, g^{1/2}(R - 2\Lambda) - \frac{1}{8\pi \ell^2} \int_{\partial M} d^3x\, h^{1/2} K, \qquad (2)$$

where

$$g \equiv \det g_{\alpha\beta}, \qquad h \equiv \det h_{ij},$$

K is the trace of the second fundamental form of the edge. Close to S the metric $g_{\alpha\beta}$ can be represented as

$$ds^2 = N^2 dt^2 + h_{ij} dx^i dx^j, \qquad (3)$$

then the second fundamental form K_{ij} is

$$K_{ij} = \frac{1}{2N} \frac{\partial h_{ij}}{\partial t}.$$

Below we shall consider the isotropic and homogeneous closed (compact without boundaries) cosmological models with Λ term without matter. In this case the metric on the 3-manifold S, i.e. at t = const, depends on the only parameter a:

$$h_{ij}(x,t) = \sigma^2 a^2(t) \tilde{h}_{ij}(x), \qquad (4)$$

where

$$\sigma^2 = \frac{4\pi\ell^2}{3}\left[\int_S d^3x\, \tilde{h}^{1/2}\right]^{-1}, \quad \tilde{h} \equiv \det \tilde{h}_{ij} .$$

The curvature for the induced metric $\tilde{h}_{ij}$ is [20]

$$^3R_{ij\kappa\ell} = k\,(\tilde{h}_{i\kappa}\tilde{h}_{j\ell} - \tilde{h}_{i\ell}\tilde{h}_{j\kappa}), \tag{5}$$

at $k = +1$, when S is a 3-sphere S^3 or a 3-sphere factorized over a descrete group (S -topologies); at $k = 0^*$, when S is a 3-torus $T^3 = S^1 \times S^1 \times S^1$ or another flat space (T -topologies); at $k = -1$, when S is a 3-hyperbolic space H^3 factorized over a descrete group (H -topologies). The space of metrics (3-5) determines the minisuperspace.

The action (2) for the metric (5) has the form

$$I_k[a] = \frac{1}{2}\int dt\left(\frac{N}{a}\right)\left(-\left(\frac{a\dot{a}}{N}\right)^2 - k a^2 + \lambda a^4\right), \tag{6}$$

where

$$\dot{a} \equiv \frac{da}{dt}, \qquad \lambda = \frac{\Lambda\sigma^2}{3} \equiv H^2.$$

In the integral of the wave function

$$\Psi_k(a_0) = \int d[a]\exp(-I_k[a]), \tag{7}$$

integration is made over all $a(t)$ which accept the value a_0 on S .

Using these expressions one can estimate in the quasiclassical approximation the Universe creation probability with $k = 0, \pm 1$. First of all one is to obtain those solutions

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*At $k = 0$ we require the condition $\int_S d^3x\,\tilde{h}^{1/2} = 1$ to be fulfilled.

of Einstein equations which can contribute into the wave function.

3. Isotropic Pseudoinstantons

From the first sight, the estimation of the wave function $\Psi_K(a_0)$ can be proceeded as follows. For the given value of Λ one finds compact gravitational instantons for the metric $g_{\alpha\beta}$ in the form of (3). For example, it is known that at $\Lambda > 0$ S^4 with radius $(\Lambda/3)^{-1/2}$ [18] is the instanton--type solution with the metric (3). Further, if it appears that the metric (3-5) cannot be attributed to the instanton found, then one may expect that in the quasiclassical approximation the corresponding wave function must be equal to zero.

Actually this is not always the right procedure, so far as while taking the integral in (7), it becomes necessary to account for real, i.e. Euclidean as well as for complex solutions (c.f. with the integral calculations in the steepest--descents approximation, when any complex extremum is taken into account). Though the physical meaning of the latters is not clear, nevertheless they do contribute to the wave function.

The extremal point for (7) is found from the Einstein equation written for the metric (3):

$$\left(\frac{\dot{a}}{N}\right)^2 + H^2 a^2 = k \, , \qquad (8)$$

under condition, that there exist such t_1 and t_2 that

$$a(t_1) = 0; \quad a(t_2) = a_0. \qquad t_1 < t_2; \tag{9}$$

The results of calculations can be represented in the form:

at $\Lambda = 0$

	$k = 0$	$k = +1$	$k = -1$
	$a(t) = \text{const}$	$a(t) = \pm \zeta(t)$ $\zeta(t) = \zeta(0) + \int_0^t d_2 N(i)$	$a(t) = \pm i\zeta(t)$
	$N = \begin{cases} 1 \\ -i \end{cases}$	$N = 1$	$N = -i$
	$dt^2 + d\Omega_0^2$ $-dt^2 + d\Omega_0^2$	$dt^2 + \zeta^2 d\Omega_1^2$ $= d\theta^2 + d\Omega_0^2$	$-dt^2 + \zeta^2 d\Omega_1^2$ $= -d\theta^2 + d\Omega_0^2$
	R^4 *	R^4	$H^3 \times R$
	R^4/Γ, T^4	R^4/Γ	$R \times (H^3/\Gamma)$

(Γ is a discrete group);

at $\Lambda > 0$:

	$k = 0$	$k = +1$	$k = -1$
	$a(t) = \frac{\text{const}}{H} e^{\pm H\zeta(t)}$	$a(t) = \pm \frac{1}{H} \sin(H\zeta(t))$	$a(t) = \pm \frac{i}{H} \sin(H\zeta(t))$
	$N = -i$	$N = \begin{cases} 1 \\ -i \end{cases}$	$N = -i$
	$-dt^2 + \frac{e^{\pm 2H\zeta}}{H^2}$	$\frac{1}{H^2}(d\theta^2 + \sin^2\theta \, d\Omega_1^2)$	$-dt^2 + \frac{\text{ch}\, Ht}{H^2} d\Omega_1^2$

*Noncompact solutions are also given here.

non-complete de Sitter	S^4 $R \times S^3$	$R_+ \times R^3$

at $\Lambda < 0$:

	$k = 0$	$k = +1$	$k = -1$
	$a(t) = \frac{const}{H} e^{\pm Ht}$	$a(t) = \frac{1}{H} sh(Ht)$	$a(t) = \begin{cases} \frac{1}{H}\sin(Ht) \\ \frac{1}{H} ch(Ht) \end{cases}$
	$N = 1$	$N = 1$	$N = \begin{cases} -i \\ 1 \end{cases}$
	$dt^2 + \frac{e^{\pm 2Ht}}{H^2} d\Omega_0^2$	$dt^2 + \frac{1}{H^2} sh^2(Ht) d\Omega_1^2$	$-dt^2 + \frac{\sin^2(Ht)}{H^2} d\Omega_{-1}^2$ $dt^2 + \frac{ch^2(Ht)}{H^2} d\Omega_{-1}^2$
	R^4	$R \times S^3$	$S \times R^3$ $R \times (H^3/\Gamma)$

The topologies given in the table are incomplete and correspond, strictly speaking, only to real solutions.

Let us discuss the topological and geometrical properties of some solutions in more details, namely at $\Lambda > 0$, $k = +1$;

$$a(t) = \pm \frac{1}{H} \sin (H \zeta(t)) .$$

When $N(t) = 1$ and $\zeta(0) = 0$, at the + sign we have

$$a(t) = \frac{1}{H} \sin (Ht) ,$$

which is just the Euclidean instanton solution S^4 .

And when $N(t) = -i$, $\zeta(0) = \frac{\pi}{2}$, we have

$$a(t)=\frac{1}{H}\,\mathrm{ch}\,(Ht)\ ,$$

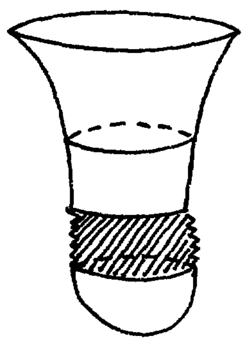
- de Sitter's solution which, however, **does not satisfy the** condition (9).

The solution is of much interest at

$$N(t)=\begin{cases} 1\ , & 0\leqslant t\leqslant \frac{\pi}{2H}-\varepsilon\ ,\\ -i\ , & \frac{\pi}{2H}+\varepsilon\leqslant t\ ,\end{cases}$$

where $0<\varepsilon\ll 1$ and the function $N(t)$ is continuously defined within the interval*

$$t\in\left[\frac{\pi}{2H}-\varepsilon\ ,\ \frac{\pi}{2H}+\varepsilon\right].$$



In fact, this solution describes the birth of the Universe from "nothing", turning from an Euclidean half-sphere into the stage of de Sitter expansion (at $Ha_0>1$ just it makes the main contribution in $\Psi_{+1}(a_0)$).

It is interesting to note that among the obtained solutions there are also exotic ones. Thus, at $N(t)=-1$, $\xi(0)=0$ ($k=+1$) we have just the same 4-sphere, but with internal time

$$d\tau=N(t)\,dt=-dt\,,$$

i.e. reversal to the coordinate time. It means that the test

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*It can be shown that such functions do exist.

particle at N = 1 is an antiparticle to that of N = -1. In this connection one should remember the recent discussion by Hawking and Page [13,21] of the key problem of time arrow in cosmology.

Analogous solutions exist for other values of k and Λ.

4. Wave Functions at Different Topologies

Now let us calculate the wave functions for the solutions presented above. First of all we calculate an action with boundary conditions

$$\begin{aligned} a(t_1) &= 0, \\ a(t_2) &= a_0 . \end{aligned} \tag{10}$$

Consider the action at $H^2 > 0$. Rewriting eq.(6) in the form

$$I_K[a_0] = \frac{1}{2}\int_{t_1}^{t_2} dt\, Na\left[-\left(\frac{\dot{a}}{N}\right)^2 - k + H^2 a^2\right], \tag{11}$$

and using (8), we have

$$I_K[a_0] = \frac{1}{2}\int_{t_1}^{t_2} dt\, Na\left[-2\left(\frac{\dot{a}}{N}\right)^2\right] = -\int_0^{a_0} da\, a\left(\frac{\dot{a}}{N}\right) \tag{12}$$

$$= -\int_0^{a_0} da\, a\,(k - H^2 a^2)^{1/2} = \frac{1}{2H^2}\int_k^{k - H^2 a_0^2} dz\, z^{1/2} .$$

As it is seen from eqs.(10) and (11), the action $I_0[a_0]$ at $k = 0$ takes two values

$$I_0^{\pm}[a_0] = \pm i \frac{Ha_0^3}{3},$$

whence it follows that the wave function

$$\Psi_0(a_0) \simeq e^{-I_0^+[a_0]} + e^{-I_0^-[a_0]} \approx \cos\left[\frac{Ha_0^3}{3}\right].$$

To calculate $\Psi_k(a_0)$ at $k \neq 0$ we proceed from the following representation (for more details see ref.[7])

$$\Psi_k(a_0) \simeq \frac{1}{2\pi i}\int_C dp\, e^{\frac{H}{3}pa_0^3}\phi_k(p),$$

where

$$p = \frac{1}{NH}\frac{\dot{a}}{a},$$

and C is a contour in the complex plane p which is parallel to the imaginary coordinate axis and lies to the right from all other singularities of the function $\phi_k(p)$:

$$\phi_k(p) = \int d[a]\exp\left(-\tilde{I}_k[a]\right),$$

$$\tilde{I}_k[a] = \frac{H}{3}pa_0^3 + I_k[a_0]. \qquad (13)$$

Note the differences concerning the wave function $\Psi_k(a_0)$ and the wave function in momentum representation $\phi_k(p)$. In the first case, when describing the birth from "nothing", a

varies from $a = 0$ to $a = a_0$ at $k = \pm 1$, while the Universe momentum reaches p when being born from different states: from $p_0 = \pm\infty$ at $k = 1$ and from $p_0 = \pm i\infty$ at $k = -1$.

From (13) one can obtain

$$\phi_k^{\pm}(p) = \int d[a] \exp\left(-\tilde{I}_k^{\pm}[a]\right),$$

where

$$\tilde{I}_k^{\pm}[a] = \pm \frac{\sqrt{k}}{3H^2} + \frac{\sqrt{k} \cdot p}{3H^2\sqrt{1+p^2}},$$

$$\sqrt{k} = \begin{cases} 1, & k = 1, \\ i, & k = -1. \end{cases}$$

and the $\pm$ sign corresponds to the sign of p_0 .

Then the wave function can be rewritten as

$$\Psi_k(a_0) \simeq \frac{1}{2\pi i} \int_C dp\, e^{\frac{H}{3} p a_0^3} \left\{ \phi_k^+(p) + \phi_k^-(p) \right\}.$$

For each of integrals in this expression, after integration in the steepest-descents approximation (c.f. [7]), we obtain

$$\frac{1}{2\pi i} \int_C dp\, e^{\frac{H}{3} p a_0^3 - \frac{p}{3H^2\sqrt{1+p^2}}} \approx \begin{cases} e^{-\frac{(1-H^2 a_0^2)^{3/2}}{3H^2}}, & H a_0 < 1, \\ \cos\left[\frac{(H^2 a_0^2 - 1)^{3/2}}{3H^2} - \frac{\pi}{4}\right], & H a_0 > 1. \end{cases}$$

$$\frac{1}{2\pi i}\int_C dp\, e^{\frac{H}{3}pa_0^3 - \frac{ip}{3H^2\sqrt{1+p^2}}} \simeq \cos\left[\frac{(H^2a_0^2+1)}{3H^2} + \frac{\pi}{4}\right],$$

whence for $\Psi_{+1}(a_0)$ we obtain

$$\Psi_{+1}(a_0) \simeq e^{\frac{1}{3H^2}}\cdot e^{-\frac{(1-H^2a_0^2)^{3/2}}{3H^2}} + e^{-\frac{1}{3H^2}}\cdot e^{-\frac{(1-H^2a_0^2)^{3/2}}{3H^2}}$$

$$\simeq e^{\frac{1}{3H^2}}\cdot e^{-\frac{(1-H^2a_0^2)^{3/2}}{3H^2}}, \qquad Ha_0 < 1 .$$

Analogously, at $Ha_0 > 1$ we have

$$\Psi_{+1}(a_0) \simeq e^{\frac{1}{3H^2}}\cos\left[\frac{(H^2a_0^2-1)^{3/2}}{3H^2} - \frac{\pi}{4}\right].$$

Now it becomes clear why at $H^2 > 0$ and $k = 1$ the probability of having the Universe with a_0 such that $Ha_0 > 1$, is not equal to zero despite that S^4 with radius H^{-1} cannot include S^3 with radius a_0. The probability is non-zero so far as a complex solution-pseudo-instanton exists. Finally, we have:

$$\Psi_0(a_0) \approx \cos\left[\frac{Ha_0^3}{3}\right],$$

$$\Psi_{+1}(a_0) = \begin{cases} e^{\frac{1}{3H^2}}\cdot e^{-\frac{(1-H^2a_0^2)^{3/2}}{3H^2}}, & Ha_0 < 1, \\ e^{\frac{1}{3H^2}}\cos\left[\frac{(H^2a_0^2-1)^{3/2}}{3H^2} - \frac{\pi}{4}\right], & Ha_0 > 1 \end{cases}$$

$$\Psi_{-1}(a_0) \simeq \cos\left(\frac{1}{3H^2}\right)\cos\left[\frac{(H^2a_0^2+1)^{3/2}}{3H^2} + \frac{\pi}{4}\right] . \qquad (14)$$

At $Ha_0 \gg 1$ the wave functions have the form

$$\Psi_0(a_0) \simeq \cos\left(\frac{Ha_0^3}{3}\right),$$

$$\Psi_{+1}(a_0) \simeq e^{\frac{1}{3H^2}} \cos\left(\frac{Ha_0^3}{3}\right), \qquad (15)$$

$$\Psi_{-1}(a_0) \simeq \cos\left(\frac{1}{3H^2}\right)\cos\left(\frac{Ha_0^3}{3}\right),$$

whence the probability ratios are found

$$|\Psi_{-1}|^2 : |\Psi_0|^2 : |\Psi_{+1}|^2 \simeq \cos\left(\frac{1}{3H^2}\right) : 1 : e^{\frac{1}{3H^2}}$$

or

$$|\Psi_{-1}|^2 \leq |\Psi_0|^2 < |\Psi_{+1}|^2 , \qquad (16)$$

i.e. the probability of having a sphere is the highest. This inequality remains unchanged at $Ha_0 \ll 1$, $H^2 \gg 1$.

The probability of birth of an inflationary Universe can also be estimated*. As it is known, [4] , one of the necessary conditions for inflation is the large value of massive scalar field $m^2\varphi^2 \gg 1$. So far as during this stage the field evolves slowly - $\dot{\varphi}/\varphi \ll H$ and, consequently, the

*Neglecting the quantum creation of matter [22].

role of H^2 plays $m^2\varphi^2$, the following can be obtained at $H^2 \gg 1$:

$$|\Psi_{-1}|^2 : |\Psi_0|^2 : |\Psi_{+1}|^2 \approx 1:1:1,$$

i.e. birth of the Universe with S, T, H -topologies is equally probable.

5. The Probability of Changing of the Universe Topology

Thus, we determine the probability of the Universe quantum creation with different topologies from "nothing", i.e. transition from $a = 0$ to a_0. Can one hence make unambiguous conclusion on topology of the modern Universe? Certainly not, since the Universe topology could have been changed during its evolution. The classical theory forbids transitions with changing of topology [23], but there are no restrictions within the quantum theory.

Now we shall find the probability for a quantum variation of topology for a "toy" model. To do this one must expand the above used minisuperspace representation.

Let us consider the case when $H^2 = 0$. Then in quasiclassical approximation the main contribution to the wave function (1) will give the Euclidean 4-torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$ with metric

$$ds^2 = 8\pi G^2 L^2 [d\theta_0^2 + d\theta_1^2 + d\theta_2^2 + d\theta_3^2], \quad 0 \leq \theta_i < 1$$

where L is a dimensionless constant much greater than unity.

This torus can contain any 3-sphere of radius $R < \frac{L}{2}$ (in $\sqrt{8\pi}\,\ell$ units) in the vicinity of which the metric reads as

$$ds^2 = 8\pi\ell^2 L^2 [dR^2 + R^2[d\alpha_1^2 + \sin^2\alpha_1 (d\alpha_2^2 + \sin^2\alpha_2 d\alpha_3^2)]],$$

$$R = const < \frac{L}{2} .$$

The same 4-torus can contain a 3-torus with metric

$$ds^2 = 8\pi\ell^2 L^2 [d\chi^2 + \chi^2 d\varphi^2 + d\theta_2^2 + d\theta_3^2],$$

$$\chi = 1, \qquad 0 \leqslant \varphi < 1 .$$

Since we are considering spaces without matter and $H^2 = 0$, only the last term will contribute into the action (2) (T^4 is an instanton at $H^2 = 0$):

$$I = -\frac{L^2}{8\pi\ell^2} \int_{\partial M} d^3\theta\, h^{1/2} K ,$$

where ∂M with account of the torus' and sphere's orientation is equal to

$$\partial M = T^3 - S^3 .$$

Then, using the relation

$$\int_S d^3x\, h^{1/2} K = \partial_n \int_S d^3x\, h^{1/2},$$

where n is unity normal vector to S, one can calculate the required action:

$$I = -\frac{L^2}{8\pi\ell^2}\int_{T^3} d^3\theta h^{1/2} K + \frac{L^2}{8\pi\ell^2}\int_{S^3} d^3\theta h^{1/2} K$$

$$= -L^2 \partial_\chi \left[\chi \int_{T^3} d\varphi\, d\theta_2\, d\theta_3\right] + L^2 \partial_R \left[R^2 \int_{S^3} d\alpha_1\, d\alpha_2\, d\alpha_3 \sin^2\alpha_2 \sin\alpha_3\right]$$

$$= -(1 - 6\pi^2 R^2) L^2.$$

Hence, the amplitude of transition of the sphere with radius R_0 into a torus is equal to

$$\Psi(S^3 \to T^3) \approx e^{-I} = e^{(1-6\pi^2 R_0^2)L^2},$$

and, as is easily seen, the corresponding amplitude of transition of the sphere with radius R_0 into a sphere with radius R_1 is equal to

$$\Psi(S^3 \to S^3) \approx e^{6\pi^2(-R_0^2 + R_1^2)L^2}$$

Therefore the ratio of probabilities of these transitions reads

$$|\Psi(S^3 \to T^3)|^2 : |\Psi(S^3 \to S^3)|^2 \approx e^{2(1-6\pi^2 R_1^2)L^2} : 1 .$$

If the sphere's radius is chosen to have the 3-sphere volume equal to that of the 3-torus

$$2\pi^2 R_1^3 = L^3,$$

one finds

$$|\Psi(S^3\rightarrow T^3)|^2:|\Psi(S^3\rightarrow S^3)|^2\simeq e^{[2-6L^2(2\pi^2)^{1/3}]L^2}\approx e^{-16L^4} \quad (17)$$

i.e. changing of topology (sphere-torus) in the considered problem is practically improbable.

There is another possibility for a sphere-torus transition, viz. annihilation of the initial sphere and creation of new torus or sphere (in a 4-torus). As it is easily seen, the final result (17) remains true, since it does not depend on the initial torus radius.

6. Conclusion

In the present paper we considered the topological and geometrical aspects of the quantum origin of the Universe. In our opinion just such narrowing of the basic problem of the properties of the 3-manifold [7] , allowing somewhat detailed mathematical consideration, can be of special interest. This circumstance urged us to consider the problem of creating of a homogeneous, isotropic Universe withot matter, but with cosmological constant (the presence of the latter allows, in particular, to interprete some conclusions in the context of the inflationary Universe scenario).

Hawking's approach to the determination of the wave function of the Universe by an Euclidean integral over compact metrics was the basis for the investigation carried out. As it turned out, into the wave function have been contributing not only gravitational instantons, i.e. real compact solutions of Einstein's equations, but also the complex ones - "pseudoinstantons". Among the found pseudoinstantons there are solutions with interesting properties, describing continuous transition from an Euclidean half-sphere into a stage of exponential expansion, with opposite directions of internal and coordinate times (i.e. with a particle-antiparticle transition).

The calculation of functional integrals by the steepest-descents method, similar to that used by Hartle and Hawking, allowed to find in the quasiclassical approximation the probabilities for creation of the Universe (transitions from $a = 0$ into $a = a_0$) with T , S , H topologies (corresponding to $k = 0$, $k = +1$, $k = -1$). The results of calculations show that the probability of having S -topology is the highest at $Ha_0 \gg 1$, however creation of the inflationary Universe ($H^2 \gg 1$) with those topologies is equally probable.

In the last section the probability for transitions with a change of topology, known to be allowed for by the quantum theory, is obtained. It is shown that at $H^2 = 0$ the 3-sphere - 3-torus transition in a four-dimensional torus is strongly suppressed with respect to a sphere-sphere transition (with different radii).

Recently Hawking and Page [24,25] have proposed a somewhat different approach to the determination of the probability of quantum state of the Universe. This approach is based on the transition from pure state of the Universe defined by the wave function $\Psi[S, h_{ij}, \phi]$ to mixed states defined by the density matrix $\rho[S', h'_{ij}, \phi'; S'', h''_{ij}, \phi'']$.

It is interesting that the density matrix can be represented via a wave function:

$$\rho[S', h'_{ij}, \Phi'; S'', h''_{ij}, \Phi''] = \Psi[S'-S'', h'_{ij}+h''_{ij}, \Phi'+\Phi''].$$

One can easily see that the corresponding probability

$$P[S, h_{ij}, \Phi] \equiv \rho[S, h_{ij}, \Phi; S, h_{ij}, \Phi]$$

is non-zero for any S , h_{ij} and Φ . Therefore, there are no constraints on the topology, geometry and material fields of the Universe within this approach (as distinct from hat of the case of "wave function" [14,15]).

Thus, the considered problem of the geometry and topology of the Universe being born allows one to determine the probabilities of creation of one or another topology, including the quantum change of the topology. It does once more show the heuristic content of Hawking's quantum cosmology, which permits one to discuss in a scientific concept the problem of "Creation of the Heaven".

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В.Г.ГУРЗАДЯН, А.А.КОЧАРЯН
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The address for requests:
Information Department
Yerevan Physics Institute
Markaryan St., 2
Yerevan, 375036
Armenia, USSR

индекс 3624

ЕРЕВАНСКИЙ ФИЗИЧЕСКИЙ ИНСТИТУТ